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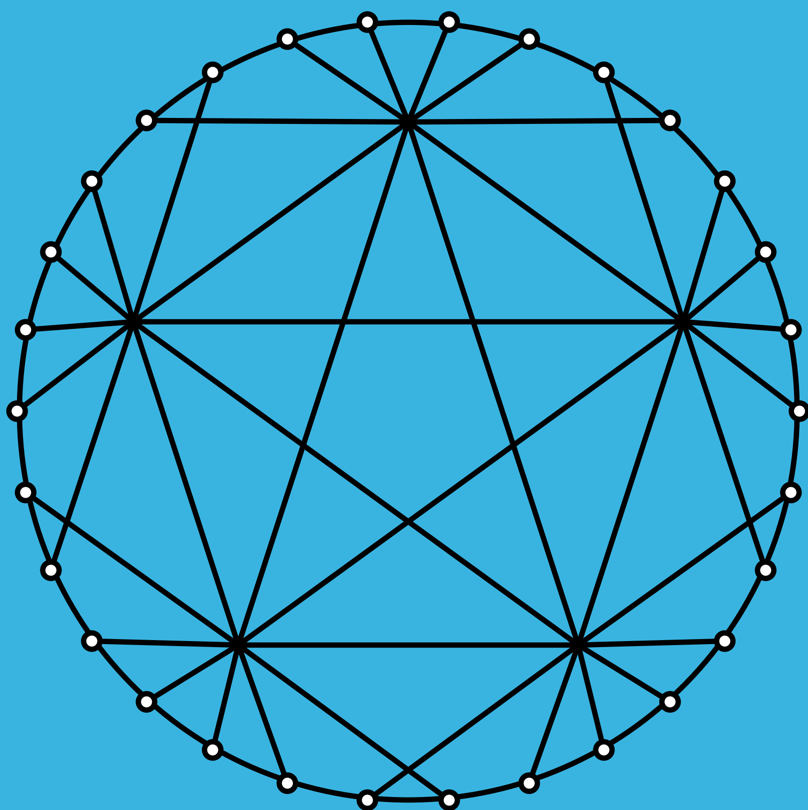

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An explicit formula for a weight enumerator of linear-congruence codes

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Abstract

An explicit formula for a weight enumerator of linear-congruence codes is provided. This extends the work of Bibak and Milenkovic [IEEE ISIT (2018) 431–435] addressing the binary case to the non-binary case. Furthermore, the extension simplifies their proof and provides a complete solution to a problem posed by them.

1 Introduction

Throughout this article, n and m denote positive integers, b denotes an integer and $\mathbb{Z}_q := \{0, 1, \ldots, q-1\} \subset \mathbb{Z}$ for a positive integer q. We will use n for a code length, m for a modulus, b for a defining parameter of a code and $\mathbb{Z}_q$ for a code alphabet.

Definition. Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. The set C of all the solutions $x = (x_1, \ldots, x_n) \in \mathbb{Z}_q^n$ for a linear congruence equation

$$a \cdot x \equiv b \pmod{m} \tag{1}$$

is said to be a *linear-congruence code* where $a \cdot x := a_1x_1 + \cdots + a_nx_n$. A linear-congruence code C is called *binary* when $q = 2$.

Key words and phrases: weight enumerator, code size, linear-congruence code, exponential sum

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Several deletion-correcting codes which have been studied are linear-congruence codes; the Varshamov-Tenengol'ts codes [8], the Levenshtein codes [7], the Helberg codes [4], the Le-Nguyen codes [5], the construction C' of Hagiwara [3] (for some parameters), the consecutively systematic encodable codes and the ternary integer codes in [2, Examples II.1 and II.5] fall into this category (Table).

Table: Examples of linear-congruence codes

Linear-congruence code[1]	q	$(a_1, \ldots, a_n)$	m	Constraints
Varshamov-Tenengol'ts code	2	$(1, \ldots, n)$	$n+1$	
Levenshtein code	2	$(1, \ldots, n)$	m	$m \geq n+1$
Helberg code[2]	2	$(v_1, \ldots, v_n)$	v_{n+1}	$s > 0$
Le-Nguyen code[3]	q	$(w_1, \ldots, w_n)$	m	$m \geq w_{n+1}$, $s > 0$
Construction C'[4]	2	$(c_1, \ldots, c_n)$	n	$b \not\equiv 0, n(n+1)/2 \pmod{n}$
Consecutively systematic encodable codes[5]	2	$(b_1, \ldots, b_n)$	2^{s+1}	$b = 0$, $s > 0$, $0 < n - s < 2^{s-1}$
Ternary integer code[6]	3	$(t_1, \ldots, t_n)$	$2^{n+1} - 1$	

The following problem concerning the size of a linear-congruence code—the number of solutions for a linear congruence equation (1)—is posed by Bibak and Milenkovic.

Problem (Bibak-Milenkovic [1])**.** Give an explicit formula for the size of a linear-congruence code.

[1] The defining parameter b for the codes in the table takes an arbitrary value unless otherwise stated.

[2] The sequence $(v_i) = (v_i(s))$ is defined by $v_i = 0$ $(i \leq 0)$ and $v_i = 1 + \sum_{j=1}^{s} v_{i-j}$ $(i \geq 1)$.

[3] The sequence $(w_i) = (w_i(q, s))$ is defined by $w_i = 0$ $(i \leq 0)$ and $w_i = 1 + (q-1) \sum_{j=1}^{s} w_{i-j}$ $(i \geq 1)$.

[4] The sequence $(c_i) = (c_i(n))$ is defined by $c_{2i-1} = i$ $(1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor)$ and $c_{2i} = n - i + 1$ $(1 \leq i \leq \lfloor \frac{n}{2} \rfloor)$.

[5] The sequence $(b_i) = (b_i(s))$ is defined by $b_i = 2^{i-1}$ $(1 \leq i \leq s)$ and $b_i = 2^{s-1} + i - s$ $(i > s)$.

[6] The sequence (t_i) is defined by $t_i = 2^i - 1$ $(i \geq 1)$.

Finding an explicit formula would be a first step toward understanding the asymptotic behavior of the size of a linear-congruence code. Bibak and Milenkovic provide a solution to the problem for the binary case. In this article, we provide a complete solution to the problem with a simple proof, which improves the argument of Bibak and Milenkovic. Actually, what we will show is how the Hamming weights of the solutions for a linear congruence equation distribute. This immediately gives an expression of the size of a linear-congruence code involving exponential sums—Weyl sums of degree one.

To state the main theorem we need notation which will be standard.

Definition. For a code $C \subseteq \mathbb{Z}_q^n$, we define a polynomial $W_C(z)$ by

$$W_C(z) := \sum_{x \in C} z^{\mathrm{wt}(x)} = \sum_{i=0}^{n} A_i(C) z^i,$$

where $\mathrm{wt}(x)$ denotes the Hamming weight and

$$A_i(C) := |\{ x \in C : \mathrm{wt}(x) = i \}| \qquad (0 \le i \le n).$$

The polynomial $W_C(z)$ is said to be the (non-homogeneous) *weight enumerator* of the code C.

Following custom due to Vinogradov in additive number theory, $e(\alpha)$ denotes $e^{2\pi\alpha\sqrt{-1}}$ for $\alpha \in \mathbb{R}$. Now we are in position to state our main theorem.

Theorem 1.1. *Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. Then the weight enumerator $W_C(z)$ of the linear-congruence code*

$$C := \{ x \in \mathbb{Z}_q^n : a \cdot x \equiv b \pmod{m} \}$$

is given by

$$W_C(z) = \frac{1}{m} \sum_{j=1}^{m} e\left(-\frac{jb}{m}\right) \prod_{i=1}^{n} \left(1 + ze\left(\frac{ja_i}{m}\right) + \cdots + ze\left(\frac{ja_i(q-1)}{m}\right)\right).$$

Corollary 1.2. *With the same notation as above, the size of the code C is given by*

$$|C| = \frac{1}{m} \sum_{j=1}^{m} e\left(-\frac{jb}{m}\right) \prod_{i=1}^{n} \left(1 + e\left(\frac{ja_i}{m}\right) + \cdots + e\left(\frac{ja_i(q-1)}{m}\right)\right).$$

2 Proof of the main theorem

The only lemma we need to prove Theorem 1.1 is the following trivial one.

Lemma 2.1.

$$\frac{1}{m}\sum_{j=1}^{m} e\left(\frac{jb}{m}\right) = \begin{cases} 1 & \text{if } b \equiv 0 \pmod{m} \\ 0 & \text{if } b \not\equiv 0 \pmod{m}. \end{cases}$$

Proof of Theorem 1.1. The proof is straightforward:

$$\begin{aligned}
&\frac{1}{m}\sum_{j=1}^{m} e\left(-\frac{jb}{m}\right)\prod_{i=1}^{n}\left(1 + ze\left(\frac{ja_i}{m}\right) + \cdots + ze\left(\frac{ja_i(q-1)}{m}\right)\right) \\
&= \frac{1}{m}\sum_{j=1}^{m} e\left(-\frac{jb}{m}\right)\prod_{i=1}^{n}\sum_{x_i\in\mathbb{Z}_q} z^{\mathrm{wt}(x_i)} e\left(\frac{ja_ix_i}{m}\right) \\
&= \frac{1}{m}\sum_{j=1}^{m} e\left(-\frac{jb}{m}\right)\sum_{(x_1,\ldots,x_n)\in\mathbb{Z}_q^n}\prod_{i=1}^{n} z^{\mathrm{wt}(x_i)} e\left(\frac{ja_ix_i}{m}\right) \\
&= \frac{1}{m}\sum_{j=1}^{m} e\left(-\frac{jb}{m}\right)\sum_{x\in\mathbb{Z}_q^n} z^{\mathrm{wt}(x)} e\left(\frac{ja\cdot x}{m}\right) \\
&= \sum_{x\in\mathbb{Z}_q^n}\left(\frac{1}{m}\sum_{j=1}^{m} e\left(\frac{j(a\cdot x - b)}{m}\right)\right) z^{\mathrm{wt}(x)} \\
&= \sum_{x\in C} z^{\mathrm{wt}(x)} \qquad \text{(By Lemma 2.1.)} \\
&= W_C(z).
\end{aligned}$$

□

Remark. The original proof by Bibak and Milenkovic [1] for the binary case uses a theorem of Lehmer [6], which states a linear congruence equation

$$a \cdot x \equiv b \pmod{m}$$

defined by $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$ has a solution $x \in \mathbb{Z}_m^n$ if and only if $\gcd(a_1, \ldots, a_n, m)$ divides b. Consequently, their result is stated depending on whether $\gcd(a_1, \ldots, a_n, m)$ divides b or not. By contrast, our result does not refer to $\gcd(a_1, \ldots, a_n, m)$ because our proof does not rely on the Lehmer theorem.

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