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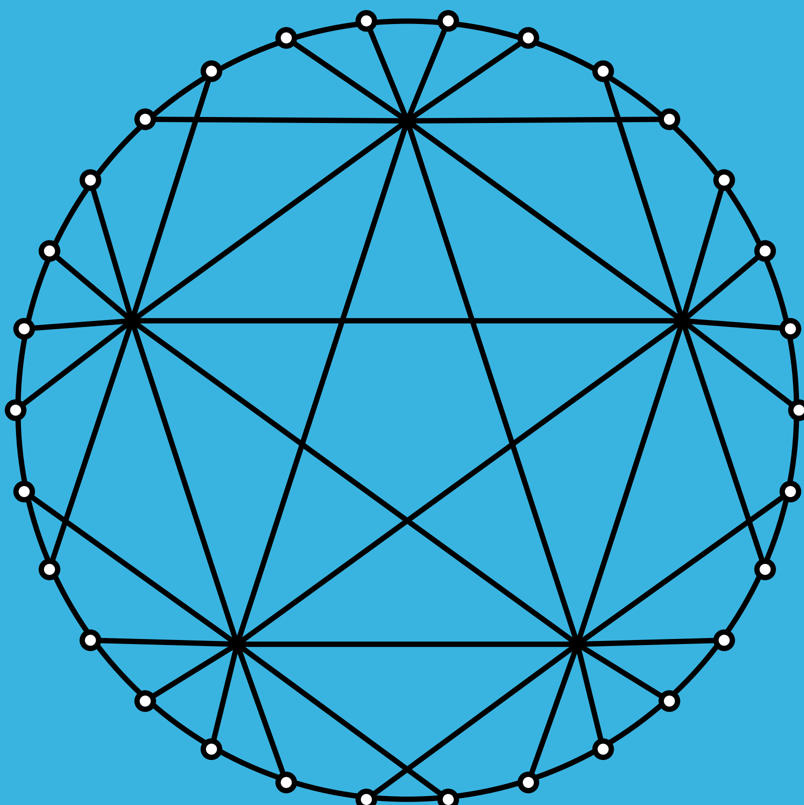

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# New symmetric 2-(71, 15, 3) designs

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**Abstract.** A symmetric 2-$(v, k, 3)$ design is called a triplane. In this paper we construct two triplanes of order twelve, i.e., symmetric 2-$(71, 15, 3)$ designs, from the binary codes of the previously known triplanes. All 146 previously known triplanes of order twelve, i.e., symmetric 2-$(71, 15, 3)$ designs, admit an action of an automorphism of order three, while these two newly constructed triplanes have full automorphism groups that are isomorphic to the elementary abelian group of order eight. Furthermore, possible actions of automorphisms of prime order on a triplane of order twelve are studied.

## 1 Introduction and preliminaries

We assume that the reader is familiar with the basic facts of design theory. For basic definitions and properties of symmetric designs not given in this paper we refer the reader to [3, 14].

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $I$ is a $t$-$(v, k, \lambda)$ *design*, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely $k$ points, and every $t$ distinct points together are incident with exactly $\lambda$ blocks. In a 2-$(v, k, \lambda)$ design each point is incident with exactly $r = \dfrac{\lambda(v-1)}{k-1}$ blocks, and $r$ is called the replication number of a design. The number $n = r - \lambda$ is called the order of a 2-$(v, k, \lambda)$ design. An isomorphism

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from one design to another is a bijective mapping from points to points and from blocks to blocks that preserves incidence. An isomorphism from a design $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted $Aut(\mathcal{D})$.

A design is called *symmetric* if it has the same number of points and blocks. If a 2-$(v,k,\lambda)$ design is symmetric, then $r = k$ and every two blocks have $\lambda$ points in common. The *dual* of a symmetric 2-$(v,k,\lambda)$ design $\mathcal{D}$, i.e., the design obtained by reversing the roles of points and blocks of $\mathcal{D}$, is also a symmetric 2-$(v,k,\lambda)$ design. The incidence matrices of a symmetric design and its dual are transpose to each other, and in general these two designs need not be isomorphic. A symmetric design that is isomorphic to its dual is called self-dual.

A symmetric 2-$(v,k,3)$ design is called a *triplane*. The known nontrivial triplanes have parameters 2-$(11,6,3)$, 2-$(15,7,3)$, 2-$(25,9,3)$, 2-$(31,10,3)$, 2-$(45,12,3)$ or 2-$(71,15,3)$. The classification of triplanes of orders three, four, six and seven is completed (see [15]), there is only one triplane of order three, exactly five triplanes of order four, 78 triplanes of order six, and 151 triplanes of order seven. Furthermore, it is known that there are at least 5421 triplanes of order nine (see [4]) and at least 146 triplanes of order twelve (see [7]). Symmetric 2-$(71,15,3)$ designs have the largest number of points among the known triplanes, since it is not known whether there exists a symmetric 2-$(81,16,3)$ design. Moreover, for many years symmetric 2-$(81,16,3)$ design has also been the smallest symmetric design for which (non)existence has not been determined (see [11]) (since 1985 when the existence of a symmetric 2-$(78,22,6)$ design was established in [13]).

All known triplanes of order twelve admit an automorphism of order three. In this paper we will give first examples of symmetric 2-$(71,15,3)$ designs which don't have this property, by constructing a triplane of order twelve whose full automorphism group is isomorphic to the elementary abelian group $E_8$. We also study possible actions of automorphisms of prime order on a triplane of order twelve. Thereby, we will prove Theorem 1.1.

**Theorem 1.1.** *There are at least* 148 *triplanes of order twelve, up to isomorphism.*

Let $\mathcal{D}$ be a triplane of order twelve and let $\alpha$ be an automorphism of $\mathcal{D}$. Let $f_\alpha$ denotes the number of fixed points of $\alpha$. For a prime $p$, $f_p$ denotes the number of fixed points of $\alpha$, $|\alpha| = p$. We will prove the following theorem.

**Theorem 1.2.** *Let $\mathcal{D}$ be a triplane of order twelve and let $\alpha$ be an automorphism of prime order acting on $\mathcal{D}$. Then $|\alpha| \in \{2, 3, 5, 7, 11\}$. Further, the following holds:*

1. $f_2 \in \{7, 9, 11, 13, 15, 17\}$,
2. $f_3 \in \{2, 5\}$,
3. $f_5 = f_7 = 1$,
4. $f_{11} = 5$.

Given a symmetric 2-$(v, k, \lambda)$ design $\mathcal{D}$, a *residual* design of $\mathcal{D}$ is the design obtained by deleting a block of $\mathcal{D}$ and keeping the points that do not belong to that block. A residual design at any block of $\mathcal{D}$ is a 2-$(v - k, k - \lambda, \lambda)$ design, so a residual design of a triplane 2-$(71, 15, 3)$ has parameters 2-$(56, 12, 3)$.

Let $\mathcal{D}$ be a 2-$(v, k, \lambda)$ design with only three distinct intersection numbers $k - r + \lambda, \rho_1$ and $\rho_2$ $(\rho_1 > \rho_2)$. Then $\mathcal{D}$ yields a strongly regular graph, called the *class graph* of $\mathcal{D}$ (see [10, Theorem 3.2.4.]). The class graph of $\mathcal{D}$ is a graph whose vertices are equivalence classes (two blocks of $\mathcal{D}$, $B_1$ and $B_2$, are equivalent if $|B_1 \cap B_2| \in \{k, k - r + \lambda\}$), where two vertices are adjacent if two blocks representing the corresponding classes have $\rho_1$ points in common.

The (linear) *code $C_{\mathbb{F}}$ of the design $\mathcal{D}$* over a finite field $\mathbb{F}$ is the vector space spanned by the incidence vectors of the blocks over $\mathbb{F}$. The vectors in $C$ are called codewords. For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{F}^n$ the number $d(x, y) = |\{i \mid 1 \leq i \leq n, x_i \neq y_i\}|$ is called a Hamming distance. An important parameter of a code $C$ is its minimum distance $d = min\{d(x, y) \mid x, y \in C, x \neq y\}$. If a linear code $C$ over a field of order $q$ is of length $n$, dimension $k$, and minimum weight $d$, then we write $[n, k, d]_q$ to show this information. For $x \in \mathbb{F}^n$ we define the weight $w(x)$ of $x$ by $w(x) = d(x, 0)$. For a linear code $C$ its minimum distance is equal to its minimum weight. The support of a nonzero vector $x = (x_1, \dots, x_n) \in \mathbb{F}^n$ is the set of indices of its nonzero coordinates, i.e. $supp(x) = \{i \mid x_i \neq 0\}$. For background reading on codes related to designs we refer the reader to [2]. In this paper, we consider binary codes of the triplanes of order twelve.

## 2 Known triplanes of order twelve

Symmetric 2-$(71, 15, 3)$ designs have the largest number of points among all known triplanes and it is not known if a triplane 2-$(v, k, 3)$ exists for $v > 71$. The first triplane of order twelve was constructed from an embeddable 2-$(56, 12, 3)$ design. More precisely, in 1980 Haemers proved the existence of at least four embeddable 2-$(56, 12, 3)$ designs and constructed four mutually non-isomorphic symmetric 2-$(71, 15, 3)$ designs that are not self-dual (see [10]). Three of the constructed triplanes have the full automorphism group of order 336 and the order of the full automorphism group of the forth triplane is 48. Together with their duals, this makes eight triplanes of order twelve constructed by Haemers. In [9], an action of an automorphism group $G \cong E_8 : F_{21}$ was presumed and nine triplanes of order twelve with that automorphism group have been constructed. Besides three pairs of dually isomorphic designs previously constructed by Haemers, that includes three new 2-$(71, 15, 3)$ designs with the full automorphism group isomorphic to $E_8 : F_{21}$ and the first example of a self-dual triplane of order twelve. With the enumeration of all triplanes of order twelve which admit an action of an automorphism of order six presented in [17] and [7], a total of 146 triplanes have been constructed so far. Among them, there are 10 self-dual designs. Let us note that triplanes with an automorphism of order six also include the triplanes previously constructed by Haemers and Garapić.

All known triplanes admit an automorphism of order three as presented in Table 1, where $G$ denotes the full automorphism group of a design.

Table 1: The full automorphism groups of known triplanes of order twelve.

| the order of $G$ | the structure of $G$ | the number of designs |
|---|---|---|
| 336 | $(E_8 : F_{21}) \times Z_2$ | 6 |
| 168 | $E_8 : F_{21}$ | 3 |
| 48 | $E_4 \times A_4$ | 26 |
| 42 | $F_{21} \times Z_2$ | 6 |
| 24 | $A_4 \times Z_2$ | 89 |
| 24 | $S_3 \times E_4$ | 16 |

**Remark 2.1.** *A residual design of a triplane of order twelve is a* 2-$(56, 12, 3)$ *design. In [10], two non-isomorphic strongly regular graphs with parameters* $(35, 16, 6, 8)$ *were constructed as class graphs of residual designs of triplanes of order twelve. In [18], it is shown that six non-isomorphic strongly regular graphs with parameters* $(35, 16, 6, 8)$ *can be constructed from residual designs of known triplanes of order twelve.*

# 3 Construction of new symmetric 2-$(71, 15, 3)$ designs

Binary and ternary codes of the known syymmetric 2-$(71, 15, 3)$ designs and their residual designs were studied in [7], where the parameters of the codes and the information on orders of their automorphism groups are given. In this section we use binary codes of known triplanes for a construction of the first example of triplane of order twelve that does not admit an action of an automorphism of order three. The method of the construction is similar to the method used in [5].

The incidence matrices of 146 known triplanes of order twelve are available at www.math.uniri.hr/∼sanjar/structures/. In the sequel we will follow the order of designs given there and denote designs by $\mathcal{D}_i$, $i \in \{1, 2, ..., 146\}$. The binary code spanned by $\mathcal{D}_i$ will be denoted by $\mathcal{C}_i$.

For every code $\mathcal{C}_i$ we construct the corresponding graph $\mathcal{G}_i$ whose vertices are codewords of $\mathcal{C}_i$ of weight 15, two vertices being adjacent if the supports of corresponding codewords share three points. Then, we are searching for cliques of size 71 in $\mathcal{G}_i$. Those cliques correspond to 2-$(71, 15, 3)$ designs. For finding cliques in graphs we used Cliquer [16]. The number of codewords of weight 15 in codes $\mathcal{C}_i$ is presented in Table 2, where $N$ denotes the quotient $\frac{|Aut(\mathcal{C}_i)|}{|Aut(\mathcal{D}_i)|}$, i.e., the number of isomorphic copies of $\mathcal{D}_i$ in $\mathcal{C}_i$. The number of vertices of $\mathcal{G}_i$ varies from 120 to 59648, and for some of the cases the construction of all cliques of size 71 is out of our reach. Those cases are marked with * in Table 2. For the cases marked with !, the number of obtained cliques is greater than $N$, and we checked if the obtained designs were previously known. For the isomorphism testing and computing the groups we used GAP [8].

From the code $\mathcal{C}_{30}$ we obtained three isomorphic copies of a design with the full automorphism group of order 8, which we denote by $\mathcal{D}$. The dual of $\mathcal{D}$ is obtained from the code $\mathcal{C}_{33}$ spanned by $\mathcal{D}_{33}$. These two designs are the first examples of a triplane of order twelve that do not admit an automorphism of order three.

Table 2: The number of codewords of weight 15 in the codes of triplanes of order 12.

| $i$ | $cw_{15}$ | N | $i$ | $cw_{15}$ | N | $i$ | $cw_{15}$ | N | $i$ | $cw_{15}$ | N |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1140 | 96 | 38 | 1270 | 1 | 75 | 161 | 1 | 112 | 3698 | 16 |
| 2 | 1044 | 2 | 39 | 1394 | 2 | 76 | 2680 | 2 | 113 | 3784 | 8 |
| 3 | 408 | 2 | 40* | 9434 | 8 | 77 | 898 | 1 | 114 | 3788 | 8 |
| 4 | 360 | 2 | 41 | 3056 | 16 | 78 | 1948 | 2 | 115 | 5174 | 1 |
| 5 | 120 | 1 | 42* | 8632 | 1 | 79 | 1948 | 2 | 116* | 16244 | 256 |
| 6 | 330 | 1 | 43* | 26538 | 128 | 80 | 946 | 1 | 117* | 22688 | 512 |
| 7 | 282 | 1 | 44 | 3580 | 16 | 81 | 2680 | 2 | 118* | 11404 | 1 |
| 8 | 192 | 1 | 45 | 2800 | 8 | 82 | 378 | 1 | 119* | 20852 | 256 |
| 9 | 386 | 1 | 46 | 1126 | 1 | 83 | 7380 | 32 | 120 | 2940 | 2 |
| 10 | 236 | 1 | 47 | 1402 | 1 | 84* | 59648 | 32768 | 121 | 1262 | 1 |
| 11 | 338 | 1 | 48 | 614 | 1 | 85 | 3068 | 8 | 122 | 926 | 1 |
| 12 | 248 | 1 | 49 | 1142 | 2 | 86 | 3452 | 8 | 123 | 4814 | 2 |
| 13 | 170 | 1 | 50 | 324 | 2 | 87* | 59648 | 32768 | 124* | 16364 | 256 |
| 14 | 314 | 1 | 51 | 348 | 2 | 88 | 7932 | 32 | 125 | 2076 | 2 |
| 15 | 386 | 1 | 52 | 1220 | 2 | 89* | 24750 | 4096 | 126* | 15788 | 256 |
| 16 | 266 | 1 | 53 | 170 | 1 | 90 | 3650 | 8 | 127 | 4778 | 2 |
| 17! | 6454 | 8 | 54 | 266 | 1 | 91 | 647 | 1 | 128 | 5594 | 2 |
| 18 | 2068 | 4 | 55 | 378 | 1 | 92 | 2736 | 16 | 129* | 14132 | 256 |
| 19! | 6550 | 8 | 56 | 306 | 1 | 93 | 2448 | 16 | 130 | 2796 | 2 |
| 20 | 2224 | 4 | 57 | 434 | 1 | 94 | 4342 | 1 | 131 | 1884 | 2 |
| 21! | 6454 | 8 | 58 | 344 | 1 | 95 | 3808 | 16 | 132 | 2700 | 2 |
| 22 | 1762 | 4 | 59 | 224 | 1 | 96 | 1154 | 1 | 133 | 5084 | 16 |
| 23! | 6550 | 8 | 60 | 314 | 1 | 97 | 3974 | 1 | 134 | 3836 | 16 |
| 24 | 2374 | 4 | 61 | 386 | 1 | 98 | 3160 | 8 | 135* | 20276 | 256 |
| 25 | 1126 | 1 | 62 | 212 | 1 | 99 | 3380 | 8 | 136 | 2856 | 2 |
| 26 | 2800 | 8 | 63 | 338 | 1 | 100 | 3670 | 1 | 137* | 23780 | 512 |
| 27 | 4540 | 16 | 64 | 272 | 1 | 101* | 19700 | 256 | 138 | 3288 | 2 |
| 28 | 1018 | 1 | 65! | 6226 | 8 | 102 | 1250 | 1 | 139 | 4366 | 1 |
| 29* | 26538 | 896 | 66 | 1762 | 4 | 103* | 20132 | 256 | 140* | 14828 | 256 |
| 30! | 5752 | 1 | 67! | 6106 | 8 | 104* | 17780 | 256 | 141 | 2700 | 2 |
| 31 | 228 | 16 | 68 | 1816 | 4 | 105 | 3068 | 8 | 142 | 3088 | 8 |
| 32 | 950 | 1 | 69! | 6226 | 8 | 106* | 21212 | 512 | 143 | 3472 | 8 |
| 33! | 6554 | 1 | 70 | 1714 | 4 | 107* | 14957 | 2 | 144 | 43338 | 4096 |
| 34 | 1270 | 1 | 71! | 6106 | 8 | 108* | 29936 | 3072 | 145 | 5486 | 8 |
| 35 | 986 | 1 | 72 | 1966 | 4 | 109 | 4144 | 16 | 146 | 710 | 1 |
| 36 | 774 | 1 | 73 | 1115 | 1 | 110* | 18668 | 512 | | | |
| 37 | 750 | 1 | 74 | 844 | 1 | 111* | 21212 | 512 | | | |

The full automorphism group $G = \langle g_1, g_2, g_3 \rangle$ of $\mathcal{D}$ is isomorphic to the elementary abelian group $E_8 = Z_2 \times Z_2 \times Z_2$. Its generators $g_1$, $g_2$ and $g_3$ are given in the sequel. Note that $g_1$, $g_2$ and $g_3$ fix seven blocks (points) of $\mathcal{D}$.

$$g_1 = (2,3)(4,5)(12,15)(13,16)(14,17)(18,21)(19,22)(20,23)(24,27)(25,28)(26,29)(30,33)(31,34)(32,35)(36,39)(37,40)(38,41)(42,45)(43,46)(44,47)(48,51)(49,52)(50,53)(54,57)(55,58)(56,59)(60,63)(61,64)(62,65)(66,69)(67,70)(68,71)$$

$$g_2 = (2,12)(3,15)(4,57)(5,54)(13,17)(14,16)(18,42)(19,49)(20,29)(21,45)(22,52)(23,26)(24,51)(25,46)(27,48)(28,43)(30,63)(31,37)(32,71)(33,60)(34,40)(35,68)(36,69)(38,65)(39,66)(41,62)(44,50)(47,53)(55,59)(56,58)(61,67)(64,70)$$

$$g_3 = (2,13)(3,16)(4,58)(5,55)(12,17)(14,15)(18,24)(19,46)(20,53)(21,27)(22,43)(23,50)(25,49)(26,44)(28,52)(29,47)(30,66)(31,61)(32,41)(33,69)(34,64)(35,38)(36,60)(37,67)(39,63)(40,70)(42,51)(45,48)(54,59)(56,57)(62,71)(65,68)$$

The orbit lengths distribution for an action of the group $G$ on $\mathcal{D}$ is $(1,2,2,2,2,2,2,2,8,8,8,8,8,8,8)$, and the representatives of the block orbits are:

$$\begin{aligned}
&1 \ldots \{1,2,3,6,7,8,9,10,11,12,13,14,15,16,17\}\\
&2 \ldots \{1,4,5,6,9,24,27,36,39,48,51,54,57,66,69\}\\
&2 \ldots \{1,4,5,7,10,25,28,31,34,49,52,55,58,61,64\}\\
&2 \ldots \{1,4,5,8,11,20,23,38,41,44,47,56,59,68,71\}\\
&2 \ldots \{6,7,8,18,19,23,26,27,28,42,43,47,48,49,53\}\\
&2 \ldots \{6,10,11,18,24,34,35,38,40,42,51,64,65,68,70\}\\
&2 \ldots \{7,9,11,19,25,32,33,36,38,43,52,62,63,66,68\}\\
&2 \ldots \{8,9,10,20,26,30,34,36,37,44,53,60,64,66,67\}\\
&8 \ldots \{1,2,3,18,24,33,39,44,46,49,53,61,62,67,68\}\\
&8 \ldots \{2,4,7,16,24,29,30,32,34,35,47,48,56,63,67\}\\
&8 \ldots \{2,4,9,15,26,28,29,37,43,45,51,59,61,65,68\}\\
&8 \ldots \{2,4,10,13,22,23,25,33,35,36,42,45,53,57,71\}\\
&8 \ldots \{2,5,6,12,20,22,32,33,37,40,47,49,55,65,66\}\\
&8 \ldots \{2,5,8,14,24,28,37,41,42,52,57,60,62,63,70\}\\
&8 \ldots \{2,5,11,17,19,21,22,23,26,39,51,58,63,64,67\}
\end{aligned}$$

**Remark 3.1.** *All residual designs of* $\mathcal{D}$ *are mutually isomorphic* $2$-$(56, 12, 3)$ *designs with intersection numbers* $0$, $2$ *and* $3$. *The corresponding class graph is a strongly regular graph with parameters* $(35, 16, 6, 8)$ *isomorphic to the graph* $G_2$ *from [18]. Residual designs of the dual of* $\mathcal{D}$ *have four intersection numbers.*

## 4 Automorphisms of symmetric 2-$(71, 15, 3)$ designs

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric $(v, k, \lambda)$ design and $G \leq \mathrm{Aut}(\mathcal{D})$. The group action of $G$ produces the same number of point and block orbits (see [14, Theorem 3.3]). We denote this number by $m$, the point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_m$, the block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_m$, and set $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. An automorphism group $G$ is said to be semi-standard if, after possible renumbering of the orbits, we have $\omega_i = \Omega_i$, for $i = 1, \ldots, m$. We denote by $\gamma_{ir}$ the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. For these numbers the following equalities hold (see [6, 12]):

$$\sum_{r=1}^{m} \gamma_{ir} = k\,, \tag{1}$$

$$\sum_{r=1}^{m} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k - \lambda)\,, \tag{2}$$

where $\delta_{ij}$ is the Kronecker delta.

**Definition 4.1.** *A* $(m \times m)$*-matrix* $(\gamma_{ir})$ *with entries satisfying conditions* (1) *and* (2) *is called an orbit matrix for the parameters* $(v, k, \lambda)$ *and orbit lengths distributions* $(\omega_1, \ldots, \omega_m)$, $(\Omega_1, \ldots, \Omega_m)$.

Hence, the *ith* row of an orbit matrix satisfies the condition (1) and the condition

$$\sum_{r=1}^{m} \frac{\Omega_i}{\omega_r} \gamma_{ir}^2 = \lambda(\Omega_i - 1) + k\,. \tag{3}$$

Let us note that for the case where an automorphism of prime order acts on a symmetric design, after a possible reordering of orbits, we have that $(\omega_1, \ldots, \omega_m) = (\Omega_1, \ldots, \Omega_m)$.

**Definition 4.2.** *An* $(s \times m)$*-matrix* $(\gamma_{ir})$*,* $s < m$*, with entries satisfying condition* (1)*, and the condition*

$$\sum_{r=1}^{m} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} \leq \lambda\Omega_j + \delta_{ij} \cdot (k-\lambda), \qquad 1 \leq i,j \leq s\,, \tag{4}$$

*is called a partial* (*block*) *orbit matrix for the parameters* $(v,k,\lambda)$ *and orbit lengths distributions* $(\omega_1, \ldots, \omega_m)$*,* $(\Omega_1, \ldots, \Omega_m)$*.*

Orbit matrices are often used in a construction of designs with a presumed automorphism group. More information on orbit matrices of block designs and a construction of 2-designs using orbit matrices can be found in [6].

In the following, we use orbit matrices to determine possible actions of automorphisms of prime order on triplanes of order twelve. The number of fixed points of an automorphism of order $p$ is denoted by $f_p$. From the following theorem it follows that $f_p \leq 18$.

**Theorem 4.3.** *[14, Corollary 3.7] Suppose that a nonidentity automorphism* $\sigma$ *of a symmetric* 2-$(v,k,\lambda)$ *design fixes* $f$ *points. Then*

$$f \leq v - 2(k-\lambda) \qquad \text{and} \qquad f \leq (\frac{\lambda}{k-\sqrt{k-\lambda}})v.$$

The second inequality in Theorem 4.3 is equivalent to $f \leq k + \sqrt{k-\lambda}$, which is easier to use. Further, according to the following theorem given in [1] we need to consider only the primes $p$ for which $p \leq k = 15$.

**Theorem 4.4.** *[1, Theorem 2.7] Let* $\mathcal{D}$ *be a symmetric* 2-$(v,k,\lambda)$ *design and* $p$ *a prime divisor of the order of* $\mathrm{Aut}(\mathcal{D})$*. Then either* $p$ *divides* $v$ *or* $p \leq k$*.*

Clearly, $f_p \equiv 71 \pmod{p}$. The following theorem gives a lower bound for the number of fixed points of an involution fixing at least one point.

**Theorem 4.5.** *[14, Proposition 4.23] Suppose that* $\mathcal{D}$ *is a nontrivial symmetric* 2-$(v,k,\lambda)$ *design, with an involution* $\sigma$*fixing* $F$ *points and blocks. If* $F \neq 0$*, then*

$$F \geq \begin{cases} 1 + \frac{k}{\lambda}, & \text{if } k \text{ and } \lambda \text{ are both even,} \\ 1 + \frac{k-1}{\lambda}, & \text{otherwise.} \end{cases}$$

It follows from Theorem 4.5 that $f_2 \geq 6$.

## 4.1 Automorphisms of prime order acting on the known symmetric 2-$(71, 15, 3)$ designs

According to [7, 17], all previously known triplanes admit an action of an automorphism of order six with one of the following orbit lengths distributions:

1. $(2, 3, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6)$,
2. $(1, 2, 2, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6)$,
3. $(1, 1, 1, 2, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6)$,

and those are the only possible actions of an automorphism of order six. Furthermore, the full automorphism groups of triplanes of order twelve admitting an automorphism of order six have one of the following orders (see Table 1): $336 = 2^4 \cdot 3 \cdot 7$, $168 = 2^3 \cdot 3 \cdot 7$, $48 = 2^4 \cdot 3$, $42 = 2 \cdot 3 \cdot 7$ and $24 = 2^3 \cdot 3$. Two newly constructed triplanes of order twelve have the full automorphism group of order $8 = 2^3$. So, there exist triplanes of order twelve having automorphisms of order two, three and seven. It is not known if there are symmetric 2-$(71, 15, 3)$ designs having an automorphism of order five, eleven or thirteen.

From given orbit lengths distributions for a group isomorphic to $Z_6$, one can see that an automorphism of order three acts on the known triplanes of order twelve with two or five fixed points, while an involution acts with seven or fifteen fixed points. Further analysis of the actions of automorphisms of order two, three and seven on the known triplanes yields the following:

- If the full automorphism group of a design is isomorphic to the one of the groups $E_8$ or $E_8 : F_{21}$, then an involution fixes seven points, and if the full automorphism group is isomorphic to the group $F_{21} \times Z_2$, then an involution fixes 15 points. In all other cases, both actions appear.
- An automorphism of order three acts with two fixed points on the ten designs with the full automorphism group isomorphic to $Z_2 \times A_4$ for which the orbit lengths distribution for $Z_6$ is given by $(2, 3, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6)$. In all the other cases, an automorphism of order three fixes five points.
- An automorphism of order seven always acts with one fixed point.

As established in [7], an automorphism of order three acting on a triplane of order twelve cannot have more than five fixed points, so $f_3 \in \{2, 5\}$. Further, we know that $6 \leq f_2 \leq 18$, hence $f_2 \in \{7, 9, 11, 13, 15, 17\}$. So far, there are no known examples for $f_2 \in \{9, 11, 13, 17\}$.

**Lemma 4.6.** *Let $\mathcal{D}$ be a triplane of order twelve and let $\alpha$ be an automorphism of $\mathcal{D}$ of order 7. Then $\alpha$ fixes exactly one point.*

*Proof.* Since $f_7 \leq 18$, we have $f_7 \in \{1, 8, 15\}$. All known actions of an automorphism of order seven on a triplane of order twelve are with one fixed point.

If $f_7 = 8$ then the cycle type of every fixed block is $1^8 7^1$ or $1^1 7^2$, i.e., a fixed block consists of eight fixed points and one $\langle\alpha\rangle$-orbit of length seven, or one fixed point and two $\langle\alpha\rangle$-orbits of length seven. Therefore, it is not possible to have more then one fixed block because the size of the intersection of two blocks must be equal to $\lambda = 3$. It follows that $f_7 \neq 8$.

If $f_7 = 15$ then the cycle type of a fixed block is $1^{15}$, $1^8 7^1$ or $1^1 7^2$. We have to consider only blocks of type $1^8 7^1$, because a fixed block of type $1^{15}$ or $1^1 7^2$ cannot intersect with other fixed blocks in $\lambda = 3$ points. The corresponding orbit matrix $OM$ is a $23 \times 23$ matrix $(\gamma_{ir})$ with entries satisfying conditions (1) and (2). Without loss of generality we can assume that the first three rows (corresponding to the fixed blocks) of $OM$ are given in Table 3.

The forth block of type $1^8 7^1$ must intersect each of the previously constructed blocks in three points. Therefore, because of the intersection with the first row, the fourth row must have exactly three 1's among the first eight positions and the following seven positions consist of five 1's and two 0's. That could not happen in a way that both intersections with the second and the third row are equal to three. So, it is not possible to construct the forth row of $OM$ using a fixed block of type $1^8 7^1$, and we can exclude the case $f_7 = 15$. □

Table 3: The unique partial orbit matrix for three fixed blocks for $Z_7$ acting with 15 fixed points.

| $OM$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 7 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 7 | 0 | 0 | 0 | 0 | 0 |

Hence, we have the following statement.

**Lemma 4.7.** *For triplanes of order twelve the following holds:*

1. $f_2 \in \{7, 9, 11, 13, 15, 17\}$,
2. $f_3 \in \{2, 5\}$,
3. $f_7 = 1$.

## 4.2 Automorphisms of order 5, 11 and 13

If an automorphism of prime order acts on a symmetric 2-$(71, 15, 3)$ design, then its order belongs to the set $\{2, 3, 5, 7, 11, 13\}$. In this section we consider possible actions of an automorphism of order 5, 11 or 13, that do not act on the known triplanes of order 12.

### 4.2.1 An action of an automorphism of order 13

**Lemma 4.8.** *A* 2-$(71, 15, 3)$ *design with an automorphism of order* 13 *does not exist.*

*Proof.* The only possible orbit lengths distribution for an action of an automorphism of order 13 on the points (and blocks) of a triplane with parameters 2-$(71, 15, 3)$ is $(1, 1, 1, 1, 1, 1, 13, 13, 13, 13, 13) \sim 1^6 13^5$, and the cycle type of a fixed block is given by $1^2 13^1$. Two fixed blocks of that type cannot intersect in $\lambda = 3$ points and, therefore, an automorphism of order thirteen does not act on a triplane of order twelve. □

### 4.2.2 An action of an automorphism of order 11

**Lemma 4.9.** *Let $\mathcal{D}$ be a triplane with parameters* 2-$(71, 15, 3)$ *and let $\alpha$, $|\alpha| = 11$, be an automorphism of $\mathcal{D}$. Then $\alpha$ acts on $\mathcal{D}$ with five fixed points (blocks) and the corresponding orbit matrix $OM$ is presented in Table 4.*

*Proof.* An automorphism of order 11 could act with 5 or 16 fixed points. If $f_{11} = 5$, then a fixed block must be of the cycle type $1^4 11^1$, and without

Table 4: The unique orbit matrix for an action of an automorphism of order 11.

| $OM$ | 1 | 1 | 1 | 1 | 1 | 11 | 11 | 11 | 11 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 0 | 11 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 11 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 11 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 11 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 11 | 0 |
| 11 | 1 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 3 | 3 |
| 11 | 0 | 1 | 0 | 0 | 0 | 2 | 2 | 2 | 3 | 2 | 3 |
| 11 | 0 | 0 | 1 | 0 | 0 | 2 | 2 | 3 | 2 | 2 | 3 |
| 11 | 0 | 0 | 0 | 1 | 0 | 2 | 3 | 2 | 2 | 2 | 3 |
| 11 | 0 | 0 | 0 | 0 | 1 | 3 | 2 | 2 | 2 | 2 | 3 |
| 11 | 0 | 0 | 0 | 0 | 0 | 3 | 3 | 3 | 3 | 3 | 0 |

loss of generality we can assume that the first five rows (corresponding to the five fixed blocks) of $OM$ are as given in Table 5.

Table 5: The unique partial orbit matrix for five fixed blocks for $Z_{11}$ acting with 5 fixed points.

| $OM$ | 1 | 1 | 1 | 1 | 1 | 11 | 11 | 11 | 11 | 11 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 0 | 11 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 11 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 11 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 11 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 11 | 0 |

Further, we will construct the possible row types for non-fixed blocks. Since every par of fixed points appeared three times in the first five blocks, it follows that a non-fixed block $B$ contains at most one fixed point. From the condition (1), it follows that $\sum_{r=1}^{11} \gamma_{ir} = 15$, and from (3) we have $11 + \sum_{r=6}^{11} \gamma_{ir}^2 = 45$ when $B$ contains one fixed point, i.e., $\sum_{r=6}^{11} \gamma_{ir}^2 = 45$ when $B$ does not contain any fixed point. Moreover, if $B$ contains a fixed point, it also contains four 2's and one 3, since its intersection with all fixed blocks must be equal to three. That yield the unique row type $(1,0,0,0,0,3,3,2,2,2,2)$, where the first five components correspond to the orbits of size one. Since every fixed point appears in 15 blocks of a triplane, obtained row type must apply to the rows $6-10$ of $OM$, and the last row corresponds to a non-fixed block that does not contain any fixed point. First five components of that row are $0,0,0,0,0$, and the rest of the row must include five 3's. That yield the unique row type $(0,0,0,0,0,3,3,3,3,3,0)$ for the last row. So, the orbit matrix presented in Table 4 is, up to isomorphism, the only possible orbit matrix for an action of an automorphism of order eleven with $f_{11} = 5$.

If $f_{11} = 16$, then the cycle type of some fixed block is $1^{15}$ or $1^4 11^1$, and the first of them can appear at most once. Moreover, the fixed block of type $1^4 11^1$ can appear at most five times, since there are five orbits of length eleven and the size of the intersection of the sixth fixed block of that type with the one of the previous five fixed block of that type would be at least eleven. So, it is not possible to have 16 fixed blocks consisting of these two types. □

### 4.2.3 An action of an automorphism of order 5

**Lemma 4.10.** *Let $\mathcal{D}$ be a triplane with parameters 2-$(71, 15, 3)$ and let $\alpha$ be an automorphism of $\mathcal{D}$. If $|\alpha| = 5$, then $\alpha$ fixes exactly one point (block) of $\mathcal{D}$.*

*Proof.* From $f_5 \leq 18$ and $f_5 \equiv 71 \pmod 5$ it follows $f_5 \in \{1, 6, 11, 16\}$.
If $f_5 = 6$, then possible cycle types for a fixed block are $1^5 5^2$ and $5^3$. It is obvious that each of them can appear at most once and that two blocks of different types cannot intersect in $\lambda = 3$ points. Therefore, $f_5 \neq 6$.
If $f_5 = 11$, then possible cycle types for a fixed block are $1^{10} 5^1$, $1^5 5^2$ and $5^3$. We have to consider only blocks of type $1^5 5^2$, because a fixed block of type $1^{10} 5^1$ or $5^3$ cannot intersect with other fixed blocks in $\lambda = 3$ points. Similarly as in the proof of Lemma 4.9, we obtain the unique partial orbit matrix $OM$ for the first five rows corresponding to fixed blocks, which is presented in Table 6.

Because of the intersection with the first row, the sixth row should have three 1's at the first five positions. However, all 2-subsets of the point set $\{1, 2, 3, 4\}$ already appear three times in the first five fixed blocks so the sixth block cannot contain two points from that set. Therefore, it is not possible to construct the sixth row corresponding to a fixed block. Hence, $f_5 \neq 11$.

For $f_5 = 16$, there are four possible cycle types for fixed blocks, namely $1^{15}$, $1^{10} 5^1$, $1^5 5^2$ and $5^3$. A block of type $1^{15}$ or $1^{10} 5^1$ could appear at most once. Each block of the type $1^5 5^2$ must have two 5's at positions $17 - 28$. Therefore, such a block could appear at most five times, since the sixth block of that type must intersect one of the previously constructed blocks of the same type in five points. In a similar way, we conclude that a block of the type $5^3$ could appear at most three times. So, it is not possible to construct 16 fixed blocks. It follows that $f_5 \neq 16$. □

Table 6: The unique partial orbit matrix for 5 fixed blocks, $Z_5$ acting with 11 fixed points.

| $OM$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 5 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 5 | 0 | 0 |

**Remark 4.11.** *There are* 66091 *orbit matrices for an action of an automorphism of order five on a triplane of order twelve with one fixed point, which we constructed by the use of a computer. However, indexing of those orbit matrices is out of our reach.*

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