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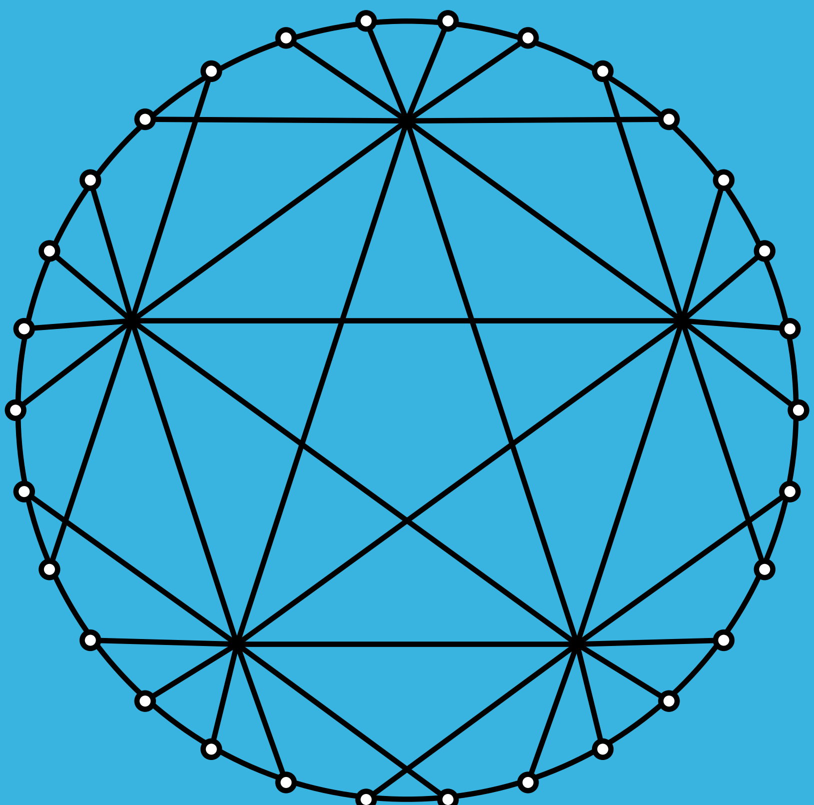

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Permanent dominance conjecture for derived partitions

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Abstract. The Permanent Dominance Conjecture is currently the most actively pursued conjecture in the theory of permanents. If A is an $n \times n$ matrix, H is a subgroup of S_n and χ is a character of H then the generalized matrix function $f_\chi(A)$ is defined as

$$f_\chi(A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If $H = S_n$ and χ is irreducible then f_χ is called an immanant. If $H = S_n$ and χ is the principal or trivial character then f_χ is called permanent. The permanent dominance conjecture states that $perA \geq \frac{f_\chi(A)}{\chi(1_n)}$ for all $A \in H_n$, where 1_n denotes the identity permutation in S_n and H_n denotes the set of all positive semidefinite Hermitian matrices. The specialization of permanent dominance conjecture to immanants has been proved true for $n \leq 13$. In this paper, we have proved that a matrix with an even number of non-positive rows and the other rows non-negative satisfies the permanent dominance conjecture. We prove that the specialization of the conjecture to immanants is satisfied by certain partitions of a natural number n. Also, we classify the partitions of a natural number n, which may not satisfy the conjecture.

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1 Introduction

The complex group algebra $\mathbb{C}(S_n)$ is the set of all functions from S_n, the symmetric group on {1,2,...,n}, to $\mathbb{C}$ endowed with the usual vector space operations and convolution multiplication. For each $\lambda \in \mathbb{C}(S_n)$ and $A = [a_{ij}] \in M_n$, the $n \times n$ complex matrices, we associate the number $f_\lambda(A)$ defined according to

$$f_\lambda(A) = \sum_{\sigma \in H} \lambda(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If A is an $n \times n$ matrix, H is a subgroup of S_n and χ is a character of H then the generalized matrix function $f_\chi(A)$ is defined as

$$f_\chi(A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If $H = S_n$ and χ is irreducible then f_χ is called an immanant. The trivial character of a matrix A is the character χ such that $\chi(\sigma) = 1$ for each $\sigma \in S_n$. In this case, f_χ is called permanent, denoted by per$A$. The alternating character of a matrix A is the character χ such that $\chi(\sigma) = 1$ when σ is an even permutation and $\chi(\sigma) = -1$ when σ is an odd permutation. In this case, f_χ is called determinant of A, denoted by $detA$.

The identity permutation in S_n is denoted by 1_n and we denote H_n, the set of positive semidefinite Hermitian matrices. The permanent dominance conjecture states that $perA \geq \frac{f_\chi(A)}{\chi(1_n)}$ for all $A \in H_n$ irrespective of the choice of χ. In other words, the permanent of a matrix dominates all the immanants. The conjecture is the permanental analogue of a result of Schur[11] which states that $detA \leq \frac{f_\chi(A)}{\chi(1_n)}$ for all $A \in H_n$.

There has been little progress made in the permanent dominance conjecture in its full generality ([10], [12]). Many authors have worked on the specialization of permanent dominance conjecture to immanants.

Pate ([3], [9]) defined a partial order $\preceq$ on the set of partitions of an integer n. Let λ and μ be two such partitions and χ and χ' be the characters associated with λ and μ by the well known bijection between partitions of n and irreducible characters of S_n. By $\lambda \preceq \mu$ we mean that $\frac{f_\chi(A)}{\chi(1_n)} \leq \frac{f_{\chi'}(A)}{\chi'(1_n)}$ for all $A \in H_n$. The specialization of permanent dominance conjecture to immanants asserts that for a natural number n, if λ is a partition of n, then $\lambda \preceq (n)$.

If $\lambda = (k, 1^{n-k})$ is a partition of a natural number n and χ_λ is the character associated with λ then χ_λ is called a single-hook immanant. Heyfron [1] proved the permanent dominance conjecture for single hook immanants and showed that $(1^n) \preceq (2, 1^{n-2}) \preceq (3, 1^{n-3}) \preceq ... \preceq (n)$.

James and Liebeck [2] proved that a partition λ of a natural number n satisfies the specialization of permanent dominance conjecture to immanants if λ has at most two parts which exceed 1. Pate [4] obtained a slightly weaker result that λ satisfies the conjecture if it has exactly two parts. Pate improved his result successively and proved that λ satisfies the conjecture if it has (i) at most two parts which exceed 2, [5] (ii) at most three parts which exceed 2 [6], (iii) at most 4 parts which exceed 2 [7], provided the second and third parts are equal in the case when there are four.

As a corollary of this last result, it has been [7] proved that the specialization of permanent dominance conjecture to immanants is true for $n \leq 13$. Pate [7] mentioned that if $n = 14$, then $(4^2, 3^2)$ is the only partition not covered by Theorem 2 in [7] and if $n = 15$ then (3^5) & $(5, 4, 3^2)$ are the only partitions not covered by Theorem 2 in [7]. I In section 3, we have shown different partitions of a natural number n which may not satisfy the conjecture.

2 Matrices which satisfy permanent dominance conjecture

In this section, we prove that all real non-negative matrices and all real non-positive matrices of even order satisfy permanent dominance conjecture. In general, matrices with an even number of non-positive rows and the other rows nonnegative satisfy the conjecture.

Theorem 2.1. *If A is either an $n \times n$ real non-negative matrix or a real non-positive matrix of even order n then* $per A \geq \frac{f_\chi(A)}{\chi(1_n)}$.

Proof. Let A be either an $n \times n$ real non-negative matrix or a real non-positive matrix of even order n. Let H be any subgroup of S_n and χ be any character. Since $\chi(1_n) = d$, a constant for a character (the degree of the representation), and $\chi(\sigma)$ is the sum of m^{th} roots of unity for each $\sigma \in S_n$, where m is the order of σ, and each root of unity is less than or equal to 1, we have $\chi(1_n) \geq \chi(\sigma)$ for each $\sigma \in S_n$.

As $\prod_{i=1}^{n} a_{i\sigma(i)}$ is non-negative for each σ and $\chi(1_n) = d$ is also non-negative,

$$\sum_{\sigma \in S_n} \chi(1_n) \prod_{i=1}^{n} a_{i\sigma(i)} \geq \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Since $\chi(1_n) = d$ is a constant, we have

$$\chi(1_n) \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \geq \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Therefore $\chi(1_n) perA \geq f_\chi(A)$. □

If A is a matrix with an even number of non-positive rows and the other rows non-negative then $perA \geq f_\chi(A)/\chi(1_n)$. In the above theorem, the positive semi-definite hermitian condition is not required though it is present in the permanent dominance conjecture. In general, real non-positive matrices may not be positive semidefinite hermitian though the permanent dominance conjecture holds for real non-positive matrices of even order.

A positive semidefinite matrix S is said to be the *correlation matrix* if each diagonal entry equal to 1.

Theorem 2.2. *Let A be a complex correlation matrix with each element having modulus 1. Then A satisfies permanent dominance conjecture.*

Proof. Let S_n be the symmetric group of order n and H is a subgroup of S_n. Let A be an $n \times n$ complex correlation matrix with $|a_{ij}| = 1$ for each $i, j = 1, 2, ...n$.

$\chi(1_n) = d$ is a constant for a character (the degree of the representation) and $\chi(\sigma)$=sum of d m^{th} roots of unity for each $\sigma \in S_n$ where m is the order of σ. Since each root of unity is less than or equal to 1 (by lexicographic ordering, the complex number $x + iy$ is less than or equal to a real number a if $x \leq a$), $\chi(1_n) \geq \chi(\sigma)$ for each $\sigma \in S_n$.

Now, since $|a_{ij}| = 1$ for each $i, j = 1, 2, ...n$ by the proposition (1, [13]) $\prod_{i=1}^{n} a_{i\sigma(i)} = 1$ for each $\sigma \in S_n$. Therefore

$$\sum_{\sigma \in S_n} \chi(1_n) \prod_{i=1}^{n} a_{i\sigma(i)} \geq \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

$$\Rightarrow \sum_{\sigma \in S_n} \chi(1_n) \prod_{i=1}^{n} a_{i\sigma(i)} \geq \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

This implies that

$$\chi(1_n) perA \geq f_\chi(A).$$ □

3 Partitions which may not satisfy the conjecture

We define a partition as a *derived partition* it is obtained by adding a non-negative integer to some of the parts of a partition or to the right of all the parts of the partition or both.

Theorem 3.1. *(Pate [7]) If λ is a partition of n with at most four parts which exceed two, it satisfies the conjecture, provided the second and third parts are equal in the case where there are four.*

Theorem 3.2. *If λ is a partition of n which cannot be derived from $(4^2, 3^2)$ or (3^5), then it satisfies the permanent dominance conjecture to immanants.*

Proof. Let λ be a partition of n which does not satisfy the specialization of permanent dominance conjecture to immanants. Then by Theorem 3.1 either it has four parts which exceed 2 with second and third parts unequal or it has five or more parts which exceed 2.Suppose it has four parts which exceed 2 with second and third parts unequal. Then it is of the form $(a, b, c, d, 2^p, 1^q)$ where $p, q \geq 0, a \geq b > c \geq d \geq 3$. Since $b \neq c, b \geq 4$. Since $a \geq b, a \geq 4$. This implies that it can be derived from $(4^2, 3^2)$.

Suppose it has five or more parts which exceed 2. Then it is of the form $(a_1, a_2, ...a_k, 2^x, 1^y)$, where $x, y \geq 0, k \geq 5, a_1 \geq a_2 \geq ... \geq 3$. This implies that it can be derived from 3^5. □

- Partition which cannot be derived from $(4^2, 3^2)$:
  $(5, 3^3, 2, 1), (5, 3^4), (4, 3^4), (5, 3^3, 2), (4^2, 2^4)$.
- Partition which cannot be derived from (3^5) :
  $(4^2, 3^2, 2), (4^3, 3, 2), (4^4, 2), (4^2, 2^4)$.
- Partition which cannot be derived from $(4^2, 3^2)$ or (3^5) :
  $(5, 3^3, 2, 1), (5, 3^3, 2), (4^2, 2^4)$.

Note that the partition $(4^2, 3^2)$ for $n = 14$, and the partitions (3^5) and $(5, 4, 3^2)$ for $n = 15$ may not satisfy the permanent dominance conjecture [7].

From the Theorem 3.2, we observed that the following partitions derived from $(4^2, 3^2)$ and (3^5) may not satisfy the conjecture:

$\boldsymbol{n = 14:}$ $(4^2, 3^2)$.

$\boldsymbol{n = 15:}$ (3^5), $(4^2, 3^2, 1)$, $(5, 4, 3^2)$, $(4^3, 3)$.

$\boldsymbol{n = 16:}$ $(3^5, 1)$, $(4, 3^4)$, $(4^2, 3^2, 1^2)$, $(5, 4, 3^2, 1)$, $(4^3, 3, 1)$, $(4^2, 3^2, 2)$, $(6, 4, 3^2)$, $(5^2, 3^2)$, $(5, 4^2, 3)$, (4^4).

$\boldsymbol{n = 17:}$ $(3^5, 1^2)$, $(4, 3^4, 1)$, $(3^5, 2)$, $(5, 3^4)$, $(4^2, 3^3)$, $(4^2, 3^2, 1^3)$, $(5, 4, 3^2, 1^2)$, $(4^3, 3, 1^2)$, $(4^2, 3^2, 2, 1)$, $(6, 4, 3^2, 1)$, $(5^2, 3^2, 1)$, $(5, 4^2, 3, 1)$, $(5, 4, 3^2, 2)$, $(4^4, 1)$, $(4^3, 3, 2)$, $(7, 4, 3^2)$, $(6, 5, 3^2)$, $(6, 4^2, 3)$, $(5^2, 4, 3)$, $(5, 4^3)$.

$\boldsymbol{n = 18:}$ $(3^5, 1^3)$, $(4, 3^4, 1^2)$, $(3^5, 2, 1)$, $(5, 3^4, 1)$, $(4^2, 3^3, 1)$, $(4, 3^4, 2)$, (3^6), $(6, 3^4)$, $(5, 4, 3^3)$, $(4^3, 3^2)$, $(4^2, 3^2, 1^4)$, $(5, 4, 3^2, 1^3)$, $(4^3, 3, 1^3)$, $(4^2, 3^2, 2, 1^2)$, $(6, 4, 3^2, 1^2)$, $(5^2, 3^2, 1^2)$, $(5, 4^2, 3, 1^2)$, $(5, 4, 3^2, 2, 1)$, $(4^4, 1^2)$, $(4^2, 3^2, 3^2)$, $(7, 4, 3^2, 1)$, $(6, 5, 3^2, 1)$, $(6, 4^2, 3, 1)$, $(6, 4, 3^2, 2)$, $(5^2, 4, 3, 1)$, $(5^2, 3^2, 2)$, $(5, 4^3, 1)$, $(5, 4^2, 3, 2)$, $(4^4, 2)$, $(4^3, 3, 2, 1)$, $(4^3, 3^2)$, $(8, 4, 3^2)$, $(7, 5, 3^2)$, $(6^2, 3^2)$, $(6, 5, 4, 3)$, $(7, 4^2, 3)$, $(6, 4^3)$, $(5^3, 3)$, $(5^2, 4^2)$.

From Theorems 3.1 and 3.2, we observe that the following partitions derived from $(4^2, 3^2)$ and (3^5) may not satisfy the conjecture:

$\boldsymbol{n = 14:}$ $(4^2, 3^2)$.

$\boldsymbol{n = 15:}$ (3^5), $(4^2, 3^2, 1)$, $(5, 4, 3^2)$.

$\boldsymbol{n = 16:}$ $(3^5, 1)$, $(4, 3^4)$, $(4^2, 3^2, 1^2)$, $(5, 4, 3^2, 1)$, $(4^2, 3^2, 2)$, $(6, 4, 3^2)$, $(5^2, 3^2)$.

$\boldsymbol{n = 17:}$ $(3^5, 1^2)$, $(4, 3^4, 1)$, $(3^5, 2)$, $(5, 3^4)$, $(4^2, 3^3)$, $(4^2, 3^2, 1^3)$, $(5, 4, 3^2, 1^2)$, $(4^2, 3^2, 2, 1)$, $(6, 4, 3^2, 1)$, $(5^2, 3^2, 1)$, $(5, 4, 3^2, 2)$, $(7, 4, 3^2)$, $(6, 5, 3^2)$, $(5^2, 4, 3)$.

$\boldsymbol{n = 18:}$ $(3^5, 1^3)$, $(4, 3^4, 1^2)$, $(3^5, 2, 1)$, $(5, 3^4, 1)$, $(4^2, 3^3, 1)$, $(4, 3^4, 2)$, (3^6), $(6, 3^4)$, $(5, 4, 3^3)$, $(4^3, 3^2)$, $(4^2, 3^2, 1^4)$, $(5, 4, 3^2, 1^3)$, $(4^2, 3^2, 2, 1^2)$, $(6, 4, 3^2, 1^2)$, $(5^2, 3^2, 1^2)$, $(5, 4, 3^2, 2, 1)$, $(4^2, 3^2, 2^2)$,

$(7,4,3^2,1)$, $(6,5,3^2,1)$, $(6,4,3^2,2)$, $(5^2,4,3,1)$, $(5^2,3^2,2)$, $(4^3,3^2)$, $(8,4,3^2)$, $(7,5,3^2)$, $(6^2,3^2)$, $(6,5,4,3)$, $(5^2,4^2)$.

Theorem 3.3. (Pate [8]) *Suppose $\alpha = (\alpha_1, \alpha_2, ..., \alpha_t)$ is a partition of n, and s is a positive integer such that $\alpha_s > \alpha_{s+1}$. Let β denote the partition $(\alpha_1, \alpha_2, ..., \alpha_s - 1, \alpha_{s+1}, ..., \alpha_t, 1)$. Then $\alpha \succeq \beta$.*

The following are ruled out cases covered under the Theorem 3.3.

$\boldsymbol{n = 15}$ **:** $(4^3,3) \succeq (4^2,3^2,1)$. Since $(4^3,3)$ satisfies the conjecture, $(4^2,3^2,1)$ also satisfies the conjecture.

$\boldsymbol{n = 16}$ **:** $(4^3,3,1) \succeq (4^2,3^2,1^2)$ and $(5,4^2,3) \succeq (5,4,3^2,1)$. Since $(4^3,3,1)$ and $(5,4^2,3)$ satisfy the conjecture, $(4^2,3^2,1^2)$ and $(5,4,3^2,1)$ also satisfy the conjecture.

$\boldsymbol{n = 17}$ **:** $(4^3,3,1^2) \succeq (4^2,3^2,1^3)$, $(5,4^2,3,1) \succeq (5,4,3^2,1^2)$, $(4^3,3,2) \succeq (4^2,3^2,2,1)$ and $(6,4^2,3) \succeq (6,4,3^2,1)$.

Since the partitions on the LHS satisfy the conjecture, the partitions on the RHS also satisfy the conjecture.

$\boldsymbol{n = 18}$ **:** $(4^3,3,1^3) \succeq (4^2,3^2,1^4)$, $(5,4^2,3,1^3) \succeq (5,4,3^2,1^3)$, $\succeq (4^3,3,2,1)$, $\succeq (4^2,3^2,2,1^2)$, $(6,4^2,3,1^2) \succeq (6,4,3^2,1^2)$, $(5,4^2,3,2) \succeq (5,4,3^2,2,1)$, $(7,4^2,3) \succeq (7,4,3^2,1)$ and $(5^3,3) \succeq (5^2,4,3,1)$.

Since the partitions on the LHS satisfy the conjecture, the partitions on the RHS also satisfy the conjecture. Finally, from the Theorems 3.1-3.3, we observed that the following partitions derived from $(4^2,3^2)$ and (3^5) may not satisfy the conjecture:

$\boldsymbol{n = 14}$ **:** $(4^2,3^2)$ (from the Theorem 3.2).

$\boldsymbol{n = 15}$ **:** (3^5), $(5,4,3^2)$ (from the Theorems 3.1, 3.2 and 3.3).

$\boldsymbol{n = 16}$ **:** $(3^5,1)$, $(4,3^4)$, $(4^2,3^2,2)$, $(6,4,3^2)$, $(5^2,3^2)$ (from the Theorems 3.1, 3.2 and 3.3).

$\boldsymbol{n = 17}$ **:** $(3^5,1^2)$, $(4,3^4,1)$, $(3^5,2)$, $(5,3^4)$, $(4^2,3^3)$, $(5^2,3^2,1)$, $(5,4,3^2,2)$, $(7,4,3^2)$, $(6,5,3^2)$, $(5^2,4,3)$ (from the Theorems 3.1, 3.2 and 3.3).

$\boldsymbol{n = 18:}$ $(3^5, 1^3)$, $(4, 3^4, 1^2)$, $(3^5, 2, 1)$, $(5, 3^4, 1)$, $(4^2, 3^3, 1)$, $(4, 3^4, 2)$, (3^6), $(6, 3^4)$, $(5, 4, 3^3)$, $(4^3, 3^2)$, $(5^2, 3^2, 1^2)$, $(4^2, 3^2, 2^2)$, $(6, 5, 3^2, 1)$, $(6, 4, 3^2, 2)$, $(5^2, 3^2, 2)$, $(8, 4, 3^2)$, $(7, 5, 3^2)$, $(6^2, 3^2)$, $(6, 5, 4, 3)$, $(5^2, 4^2)$ (from the Theorems 3.1, 3.2 and 3.3).

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