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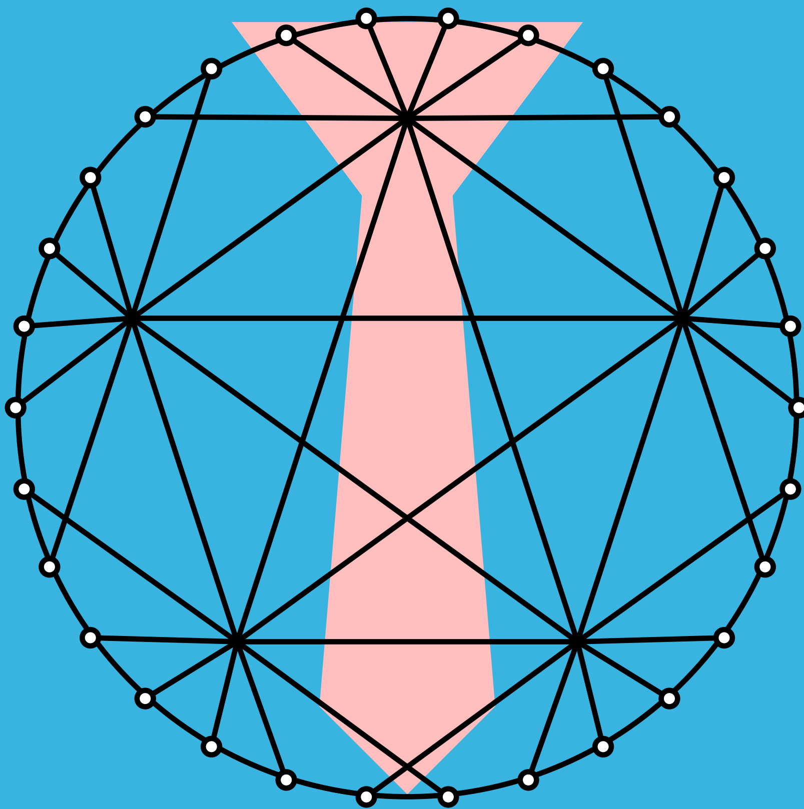

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On $\delta^{(k)}$-coloring of graph products

Merlin Thomas Ellumkalayil and Sudev Naduvath*

Abstract. An edge which is incident on two vertices that are assigned the same color is called a bad edge. A near proper coloring is a coloring that minimises the number of bad edges in a graph G, by permitting few color classes to have adjacency between the elements in it. A near proper coloring, that uses k colors where $1 \leq k \leq \chi(G) - 1$, which allows at most one color class to be a non independent set to minimise the number of bad edges resulting from the same is called a $\delta^{(k)}$-coloring. In this paper, we determine the minimum number of bad edges, $b_k(G)$, resulting from a $\delta^{(k)}$-coloring of some graph products viz. direct product of two graphs $G \times H$ and corona product of two graphs $G \circ H$, for all different possible values of k by investigating an optimal $\delta^{(k)}$-coloring that results in minimum number of bad edges.

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [1, 11, 19] and for graph classes, we refer to [2, 9]. Further, for the terminology of graph coloring, see [3, 13, 16]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors required to color a graph in such a way that if any pair of vertices receive a same color then it should be a non adjacent pair. In an improper coloring, an edge uv is a bad edge if $c(u) = c(v)$, where $c(u)$ and $c(v)$ are the colors assigned to the vertices u and v respectively. If the minimum number of colors required to color a graph properly is not available, then coloring the graph by permitting only one color class to be

*Corresponding author: sudev.nk@christuniversity.in

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a non independent set so as to minimise the number of bad edges resulting from the same is called $\delta^{(k)}$-coloring (see [15]). A handful of work on a $\delta^{(k)}$-coloring of certain graph classes can be seen in the literature. The interested reader is referred to recent articles on a $\delta^{(k)}$-coloring of graphs, see [4, 7, 6, 5] and also few engrossing studies on the concept of improper and proper coloring, see [15, 17, 18].

Definition 1.1. A coloring that permits few color classes to have adjacency between the vertices in it to minimise the number of bad edges in a graph is called a near proper coloring.

Definition 1.2. A $\delta^{(k)}$-coloring of a graph G with k available colors, where $1 \leq k \leq \chi(G) - 1$, is a near proper coloring, which minimises the number of bad edges by permitting at most one color class to have adjacency between the elements in it. The minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of G is denoted by $b_k(G)$.

Let the k available colors required for a $\delta^{(k)}$-coloring of G be $c_1, c_2 \ldots, c_k$ with their respective color classes $C_1, C_2, \ldots, C_k$, throughout the discussion. Without loss of generality, the color class C_1 is the color class that is allowed to have adjacency between the vertices in it. It is clear from Definition 1.2 that when the number of available colors k is 1, the number of bad edges resulting from $\delta^{(k)}$-coloring of any graph G is $|E(G)|$. Hence, we do not consider a $\delta^{(1)}$-coloring of any graph and thereby do not consider a $\delta^{(1)}$-coloring of bipartite graph as well. Now, following are the results obtained for a $\delta^{(k)}$-coloring of direct product and corona product of certain classes of graphs. The results focus on the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring and provides an optimal $\delta^{(k)}$-coloring that results in the same for each different values of k where $2 \leq k \leq \chi(G) - 1$. Furthermore, the concept of independence number and independence set is also used in this paper for determining the minimum number of bad edges. The readers can refer to the below definition for independent set and independence number.

Definition 1.3. A set V of vertices in a graph G is said to be independent if no two vertices in the set V are adjacent to each other. The maximum number of vertices in an independent set is called the independence number of G and it is denoted by $\alpha(G)$.

2 A $\delta^{(k)}$-coloring of direct product of graphs

The main focus of this section is to obtain the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of direct product of two graphs. Firstly, recall the definition of direct product of two graphs:

Definition 2.1. [10] In the *direct product* of two graphs, two vertices (g, h) and (g', h') are adjacent if both $gg' \in E(G)$ and $hh' \in E(H)$. The direct product of G and H is denoted by $G \times H$.

Consider two graphs G and H of order m and n respectively. In $G \times H$, there are a total of $m \times n$ vertices. Thus, there are m sets each having n vertices or n sets each having m vertices in the direct product. Throughout the discussion, it is considered that $n \geq m$ (note that, since direct product is a commutative product, all the results discussed below hold for $n < m$ as well) and that there are m set each of n vertices. The first set of n vertices is denoted as g_1h_j, where $1 \leq j \leq n$, the second set is denoted as g_2h_j, where $1 \leq j \leq n$, and so on the m-th set of n vertices is denoted as g_mh_j, where $1 \leq j \leq n$.

If either G or H is bipartite, then their direct product $G \times H$ is bipartite and hence the following discussion does not consider a $\delta^{(k)}$-coloring of bipartite graphs. This study solely focuses on cycle graph C_{2n+1} complete graph K_n. It is known that, $\chi(G \times H) \leq \min(\chi(G), \chi(H))$ (see [12]).

When $C_m \times C_n$ and $C_m \times K_n$ are considered, $\chi(C_m \times C_n) = 3$, when both n and m are odd, and $\chi(C_m \times K_n) = 3$, when m is odd. Thus, for these two cases a $\delta^{(2)}$-coloring is considered. For $K_m \times K_n$, the value of k will be $2 \leq k \leq \min\{m, n\} - 1$. The direct product is commutative and hence the study concerned focuses on either a $\delta^{(k)}$-coloring of $G \times H$ or $H \times G$. The following are the results obtained from a $\delta^{(k)}$-coloring of direct product of cycle graph and complete graph with their possible combination.

Theorem 2.2. *For $C_m \times C_n$, where m and n are odd, the number of bad edges resulting from $\delta^{(2)}$-coloring is $b_2(C_m \times C_n) = 2m$.*

Proof. For $C_m \times C_n$, $\chi(C_m \times C_n) = 3$ and it is a 4-regular graph. In this case, $k = 2$. As mentioned above, we consider the color class C_1 to be a non-independent set and hence it is clear that every other color class is an independent set. Hence, it is obvious that a $\delta^{(2)}$-coloring of a graph is based on the independence number of the graph. The independence number of $G \times H$ is given by $\alpha(G \times H) \geq \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$ (see [10]) and the independence number of direct product of odd cycles is already been discussed in [14] as $(n-1)\frac{m}{2}$. The minimum number of monochromatic edges obtained from $\delta^{(2)}$-coloring for an r-regular is $\frac{r(n-2\alpha)}{2}$ (see Theorem 2.12 of [6]). Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $C_m \times C_n$ is $b_2(C_m \times C_n) = \frac{4(mn-2(n-1)\frac{m}{2})}{2} = 2m$. □

Theorem 2.3. *For $C_m \times K_n$, where m and n are odd, the number of bad edges resulting from $\delta^{(2)}$-coloring is $b_2(C_m \times K_n) = n(n-1)$.*

Proof. It is known that, $\chi(C_m \times K_n) = 3$ and hence $k = 2$. As explained in Theorem 2.2, first the concept of independence number is used and an upper bound for the minimum number of monochromatic edges obtained from $\delta^{(2)}$-coloring is provided. It can be noted that, the independence number of $G \times H$ is $\alpha(G \times H) \geq \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$ (see [10]). Thus, in this case, $\alpha(C_m \times K_n) \geq \max\{\alpha(C_m)|V(K_n)|, \alpha(K_n)|V(C_m)|\} = \max\{n\lfloor\frac{m}{2}\rfloor, m|\}$. Since $m \leq n$, $\alpha(C_m \times K_n) \geq n\lfloor\frac{m}{2}\rfloor$. The number of bad edges resulting from $\delta^{(2)}$-coloring of a regular graph is $\frac{r(n-2\alpha)}{2}$ (see Theorem 2.12 of [6]). Thus, $b_2(C_m \times K_n) \geq \frac{r(n-2\alpha)}{2} \geq \frac{2(n-1)(mn-2n\lfloor\frac{m}{2}\rfloor)}{2} \geq n(n-1)$. Hence, $b_2(C_m \times K_n) \geq n(n-1)$.

It is to be proved first that, $b_2(C_m \times K_n) = n(n-1)$. For this, it suffices to find a $\delta^{(2)}$-coloring that results in $n(n-1)$ monochromatic edges. None of the vertices g_1h_j, where $1 \leq j \leq n$, are adjacent to each other. Hence, all these vertices can have a single color, say c_1. Each vertex g_1h_j is adjacent to every g_2h_j except for its corresponding vertex. Hence, the vertices g_2h_j can be assigned the color c_2 or c_1. However, the aim is to minimise the number of monochromatic edges and hence use the color c_2 to color g_2h_j. Third set of vertices g_3h_j can be colored with the color c_1 and the fourth set g_4h_j can be assigned the color c_2. Thus, alternatively color each n set with two colors c_1 and c_2 properly. The last set of n vertices that is g_mh_j, where $1 \leq j \leq n$, has to be given the color c_1 to maintain the condition of a $\delta^{(k)}$-coloring of graphs. The only edges that provide monochromatic edges is the edges between the first set of vertices (g_1h_j) and the m-th set of vertices (g_mh_j). Each of the n vertices in the set g_1h_j which has the color c_1 are adjacent to $n-1$ vertices of the set g_mh_j given the color c_1, which results in a situation where there are a total of $n(n-1)$ monochromatic edges. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $C_m \times K_n$ is $n(n-1)$. □

Theorem 2.4. *For $K_m \times K_n$, where m and n are odd, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is*

$$b_k(K_m \times K_n) = \frac{n(n-1)(m-k)(m-k+1)}{2}.$$

Proof. It is know that, $\chi(K_m \times K_n) = \min\{m, n\}$ (see [12]). Since it is assumed that $n \geq m$, $\chi(K_m \times K_n) = m$ and hence k can be $2 \leq k \leq m-1$. In this case, there are two possible $\delta^{(k)}$-colorings which are as explained.

In $K_m \times K_n$, except for its corresponding vertices every vertex is adjacent to every other vertex. Thus, either every corresponding vertex, which is an independent set, can be assigned a single color or every n vertices in a single set, which is also an independent set, can be given a single color. Since $2 \leq k \leq m-1$, the former coloring will lead to a situation where there are $\frac{m(m-1)(n-k)(n-k+1)}{2}$ monochromatic edges and the latter provides $\frac{n(n-1)(m-k)(m-k+1)}{2}$ monochromatic edges. Since $n \geq m$, the minimum number of monochromatic edges obtained is $\frac{n(n-1)(m-k)(m-k+1)}{2}$, when $n > m$, and both are same, when $n = m$. Thus, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $K_m \times K_n$ is $\frac{n(n-1)(m-k)(m-k+1)}{2}$, when $n \geq m$. □

3 A $\delta^{(k)}$-coloring of corona product of graphs

The corona product is not commutative and hence in this section all the possible combination of a $\delta^{(k)}$-coloring of corona product of path graph, cycle graph and complete graphs are taken into consideration.

Definition 3.1. [8] Let G be a graph on n vertices and H be another graph. The *corona product* of two graphs G and H, denoted by $G \circ H$, is obtained by taking n copies of H, and each vertex in G is adjacent to every vertex of the corresponding H. That is, every i-th vertex of G is adjacent to each vertex of i-th copy of H, where $1 \leq i \leq n$.

Throughout the section, the vertex v_i, where $1 \leq i \leq n$, corresponds the vertices of the graph G and the vertices v_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq m$, are the vertices of the i-th copy of H corresponding to v_i vertex of G. For instance, the vertices $v_{11}, v_{12}, \ldots, v_{1n}$ are the vertices of the first copy of H corresponding to the vertex v_1 in G.

Theorem 3.2. *For $P_m \circ P_n$, the number of bad edges resulting from $\delta^{(2)}$-coloring is* $b_2(P_m \circ P_n) \leq \min\{\lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor (n-1), \lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor\}$.

Proof. The corona product $P_m \circ P_n$ is 3-colorable and hence k can only be 2. There are two possible $\delta^{(2)}$-colorings as explained below. The first coloring is to color the vertices, $v_1, v_2, \ldots, v_m$, of P_m with two colors c_1 and c_2 alternatively. Thus, $c(v_{2i+1}) = c_1$, where $0 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$ and $c(v_{2i}) = c_2$, where $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$. This coloring will provide $\lceil \frac{m}{2} \rceil$ independent vertices that have the color c_1 and $\lfloor \frac{m}{2} \rfloor$ independent vertices with the color c_2.

The path graph P_n corresponding to the vertices of P_m which have the color c_1, can be alternatively assigned the color c_1 and c_2. If every first vertex of these $\lceil \frac{m}{2} \rceil$ copies of P_n is given the color c_1, the remaining vertices of each copy is alternatively colored with c_2 and c_1. This coloring will cause for a situation where there are $\lceil \frac{n}{2} \rceil$ independent vertices which have the color c_1 and $\lfloor \frac{n}{2} \rfloor$ independent vertices the color c_2 and vice versa if every first vertex of these $\lceil \frac{m}{2} \rceil$ copies of P_n is given the color c_2. The former will increase the number of monochromatic edges due to the increase in the number of vertices that receive the color c_1 and the later will decrease the same by one. Hence, every first vertex of these $\lceil \frac{m}{2} \rceil$ copies of P_n is given the color c_2, and the remaining vertices of these copies are alternatively assigned the color c_1 and c_2. Thus, there are $\lceil \frac{m}{2} \rceil$ vertices in P_m with the color c_1 which are adjacent to $\lfloor \frac{n}{2} \rfloor$ vertices of its corresponding path graph P_n whose color is c_1, which cause a scenario where there are $\lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor$ monochromatic edges between them. Now, the $\lfloor \frac{m}{2} \rfloor$ copies of P_n, corresponding to $\lfloor \frac{m}{2} \rfloor$ vertices in P_m that receive the color c_2, should solely be given the color c_1 to maintain the requirements of a $\delta^{(k)}$-coloring of graphs, which will cause for no monochromatic edge between these copies of P_n and its corresponding vertices with color c_1 in P_m. However, every edge in these copies of P_n will be a monochromatic edge, leading to a total of $\lfloor \frac{m}{2} \rfloor (n-1)$ monochromatic edges. Thus, the total number of monochromatic edges resulting from this particular $\delta^{(2)}$-coloring is $\lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor (n-1)$.

In the second $\delta^{(k)}$-coloring, begin coloring vertices of P_m alternatively with the colors c_2 and c_1. Thus, $c(v_{2i+1}) = c_2$, where $0 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$ and $c(v_{2i}) = c_1$, where $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$. This coloring will cause a situation where there are $\lceil \frac{m}{2} \rceil$ independent vertices that are assigned the color c_2 and $\lfloor \frac{m}{2} \rfloor$ independent vertices with the color c_1, which thereby yields to coloring $\lceil \frac{n}{2} \rceil$ copies of P_n solely with the color c_1, leading to $\lceil \frac{m}{2} \rceil (n-1)$ monochromatic edges. The remaining copies of P_n corresponding to $\lfloor \frac{m}{2} \rfloor$ vertices of P_m which have the color c_1, are assigned the color c_2 and c_1 alternatively leading to $\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$ monochromatic edges. Thus, the total number of monochromatic edges resulting from this $\delta^{(2)}$-coloring is $\lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$.

When both the colorings are compared, the monochromatic edges obtained from both is the same, when m is even, and is $\min\{\lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor (n-1), \lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor\}$, when m is odd. □

Theorem 3.3. *For $C_m \circ C_n$, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is*

$$b_k(C_m \circ C_n) = \begin{cases} \min\left\{\frac{n(3m-1)+4}{4}, \frac{m(n+2)}{2}\right\}, & \text{if } m \text{ is odd, } n \text{ is even and } k=2, \\ \min\left\{\frac{3n(n+1)}{4}, \frac{m(n+3)+2n}{2}\right\}, & \text{if } m \text{ is even, } n \text{ is odd and } k=2, \\ m, & \text{if } m, n \text{ are both odd and } k=3, \\ \min\left\{\frac{3mn}{4}, \frac{m(n+2)}{2}\right\}, & \text{if } m, n \text{ are both even and } k=2. \end{cases}$$

Proof. The different cases for the $\delta^{(k)}$-coloring of $C_m \circ C_n$, for different parities of m and n and for different values of k are explained as below.

Case 1: Let $k = 2$, m be odd and n be even. Let the two colors be c_1 and c_2. The odd cycle C_m will result in one monochromatic edges when colored with c_1 and c_2 (see Propoition 2.3, [15]) and an even length cycle C_n can be properly colored with two colors. As explained in Theorem 4.5 $\lfloor \frac{m}{2} \rfloor$ vertices of C_m are assigned the color c_2, its corresponding C_n's ($\lfloor \frac{m}{2} \rfloor$ in number) should be exclusively colored with c_1 to meet the requirements of a $\delta^{(k)}$-coloring of graphs. This coloring will result in a condition where there exists $\lfloor \frac{m}{2} \rfloor n$ monochromatic edges. Also, the $\lceil \frac{m}{2} \rceil$ copies of C_n are given the color c_1 and c_2 alternatively as they are adjacent to $\lceil \frac{m}{2} \rceil$ vertices of C_m which have the color c_1. This coloring will yield $\lceil \frac{m}{2} \rceil \frac{n}{2}$ monochromatic edges. Thus, the total number of monochromatic edges obtained from $\delta^{(2)}$-coloring of $C_m \circ C_n$ is $\lfloor \frac{m}{2} \rfloor n + \lceil \frac{m}{2} \rceil \frac{n}{2} + 1 = \frac{n(3m-1)+4}{4}$, when m is odd and n is even. Now, another possible $\delta^{(2)}$-coloring for this case is that, the cycle C_m is colored with a single color c_1, leading to m monochromatic edges and the m copies of C_n are alternatively assigned the color c_1 and c_2. This coloring will cause for a situation where there exists $\frac{mn}{2}$ monochromatic edges between C_m and C_n. Thus, the total number of monochromatic edges obtained from this $\delta^{(2)}$-coloring is $m + \frac{mn}{2} = \frac{m(n+2)}{2}$. Hence, the number of bad edges resulting from $\delta^{(2)}$-coloring of $C_m \circ C_n$ is $\min\{\frac{n(3m-1)+4}{4}, \frac{m(n+2)}{2}\}$, when m is odd and n is even.

Case 2: Let $k = 2$ and m be even and n be odd. Since m is even, coloring C_m with c_1 and c_2 will provide no monochromatic edges in C_m. It is known that, there are m copies of C_n out of which $\frac{m}{2}$ copies that are adjacent to the corresponding vertices of C_m which has the color c_1 can be alternatively colored with c_1 and c_2. Thus, there are $\lceil \frac{n}{2} \rceil$ vertices receiving the color c_1 and $\lfloor \frac{n}{2} \rfloor$ vertices the color c_2. This coloring provide one monochromatic edge in each of $\frac{m}{2}$ C_n's and $\frac{m}{2}\lfloor \frac{n}{2} \rfloor$ monochromatic edges between them. The remaining $\frac{m}{2}$ copies of C_n's that are adjacent to the vertices of C_m ($\lfloor \frac{m}{2} \rfloor$ vertices) which are assigned the color c_2, are solely colored with the color c_1, to meet the requirements of a $\delta^{(k)}$-coloring of graphs. This coloring will thereby result in $\lfloor \frac{m}{2} \rfloor n$ monochromatic edges. Thus, the total number of monochromatic edges resulting from this $\delta^{(2)}$-coloring is $\frac{m}{2}\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor n + \frac{m}{2} = \frac{3n}{2}(\frac{n+1}{2})$. Now, the second possible $\delta^{(2)}$-coloring for this case is same as that of second $\delta^{(2)}$-coloring explained in Case 1 mentioned above. This coloring will lead to all the edges in C_m to be monochromatic (since all the m vertices are assigned the color c_1). However, since n is odd, there will be one monochromatic edge in each of the n copies of C_n and $m\lceil \frac{n}{2} \rceil$ monochromatic edges between C_n and C_m. Thus, the total number of monochromatic edges in this case is $\frac{m(n+3)+2n}{2}$. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $C_m \circ C_n$ is $\min\{\frac{3n(n+1)}{4}, \frac{m(n+3)+2n}{2}\}$, when m is even and n is odd.

Case 3: Let $k = 3$ and both m and n be odd. It can be noted that, $\chi(C_m \circ C_n) = 4$, when m and n are odd, and hence $k = 2$ and $k = 3$. Firstly, the $\delta^{(3)}$-coloring of $C_m \circ C_n$ is discussed as follows. Since $k = 3$, maximise the use of the colors c_2 and c_3 and minimise the use of color c_1 as much as possible. A $\delta^{(3)}$-coloring that exactly explains the same is as follows. Assign the vertices of C_m alternatively with the colors c_2 and c_3 and the last vertex v_m is assigned the color c_1. This is a proper coloring of an odd cycle with three colors. Each of the $m-1$ copies of C_n corresponding to the vertices of the C_m, whose colors are c_2 and c_3, can be given the colors c_1 and c_3, and c_1 and c_2 respectively. This coloring will cause for a scenario where there exist one monochromatic edge in each of the $m-1$ copies of C_n. The corresponding C_n of the m-th vertex of C_m that is assigned the color c_1 can be properly colored with three colors, leading to one monochromatic edge between this vertex and the m-th copy of C_n. Thus, the $\delta^{(3)}$-number is m, when m and n are odd.

Case 4: Let $k = 2$ and both m and n be odd. In this case, there are two possible $\delta^{(2)}$-colorings as explained in Case 1 and Case 2. The first $\delta^{(2)}$-coloring is obtained by alternatively coloring C_m with two colors leading to one monochromatic edge in C_m. The $\lceil \frac{m}{2} \rceil$ copies of C_ns, corresponding to the $\lceil \frac{m}{2} \rceil$ vertices of C_m that are colored with c_1 are alternatively given the color c_1 and c_2 and $\lfloor \frac{m}{2} \rfloor$ copies of C_n corresponding to $\lfloor \frac{m}{2} \rfloor$ vertices are assigned the color c_2, are assigned the color vertices of C_m which have the color c_2 are exclusively given the color c_1 to meet the prerequisites of $\delta^{(k)}$-coloring of graphs. This coloring will have a total of $1+\lceil\frac{m}{2}\rceil+n\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil\lceil\frac{n}{2}\rceil = \frac{3m(1+n)-n+7}{4}$ monochromatic edges in $C_m \circ C_n$. The second $\delta^{(2)}$-coloring is same as the second $\delta^{(k)}$-coloring of Case 1. The cycle C_m is exclusively colored with c_1 and the corresponding C_n's are assigned the color c_1 and c_2. This $\delta^{(2)}$-coloring will have $m+m+m\lceil\frac{n}{2}\rceil = \frac{m(n+5)}{2}$ monochromatic edges. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $C_m \circ C_n$ is $\min\{\frac{3m(1+n)-n+7}{4}, \frac{m(n+5)}{2}\}$, when m and n are odd.

Case 5: Let $k=2$ and m and n be even. It is to be noted that, $\chi(C_m \circ C_n) = 3$, when both m and n are even. Hence, $k=2$. There are two possible $\delta^{(2)}$-colorings in this case which are as discussed below:

In the first $\delta^{(2)}$-coloring, the cycle C_m can be properly colored with two colors. Each of the $\frac{m}{2}$ copies of C_n are alternatively given the colors c_1 and c_2 as they are adjacent to $\frac{m}{2}$ vertices of C_m which have the color c_1, leading to $\frac{mn}{4}$ monochromatic edges between them. The remaining $\frac{m}{2}$ copies of C_n is solely assigned the color c_2 to maintain the requirements of a $\delta^{(k)}$-coloring of graphs. Thus, this coloring will result in a situation where there are $\frac{mn}{2}$ monochromatic edges. Hence, the total number of minimum monochromatic edges obtained from this $\delta^{(2)}$-coloring is $\frac{3mn}{4}$. Now, the second $\delta^{(2)}$-coloring is same as that of the second $\delta^{(2)}$-coloring explained in Case 1 and the number of monochromatic edges obtained from this case is $\frac{m(n+2)}{2}$. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $C_m \circ C_n$ is $\min\{\frac{3mn}{4}, \frac{m(n+2)}{2}\}$, when m and n are even. This completes the proof. □

Theorem 3.4. *For $P_m \circ C_n$, the the number of bad edges resulting from $\delta^{(2)}$-coloring is*

$$b_2(P_m \circ C_n) \leq \begin{cases} \frac{3mn}{4}, & \text{if both } m \text{ and } n \text{ are even,} \\ \frac{3mn-n}{4}, & \text{if } m \text{ is odd and } n \text{ is even,} \\ \frac{3m(n+1)}{4}, & \text{if } m \text{ is even and } n \text{ is odd,} \\ \frac{3(m+nm+1)-n}{4}, & \text{if both } m \text{ and } n \text{ are odd.} \end{cases}$$

Proof. It is to be noted that, $\chi(P_m \circ C_n) = 3$ and hence, $k=2$. For different parities of m and n, different possible $\delta^{(2)}$-colorings and the number of monochromatic edges obtained from the same is as explained below.

Case 1: Let both m and n be even. In this particular case, coloring P_m alternatively with c_1 and c_2 will have no monochromatic edges in P_m. However, the $\frac{m}{2}$ copies of C_n are alternatively assigned the color c_1 and c_2 and the remaining $\frac{m}{2}$ copies of C_n will only have the color c_1 in order to maintain the requirements of a $\delta^{(k)}$-coloring of graphs. Thus, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $P_m \circ C_n$ is $\frac{m}{2}\frac{n}{2} + \frac{m}{2}n = \frac{3mn}{4}$, when m and n are even.

Case 2: Let m be odd and n be even. As explained in Theorem 4.5, there can be two possible $\delta^{(2)}$-colorings for this case. The first coloring is when P_m is alternatively given the colors c_1 and c_2 and the second one the vertices of P_m is assigned the colors c_2 and c_1 alternatively. The former will cause a situation where there are $\lceil \frac{m}{2} \rceil \frac{n}{2} + \lfloor \frac{m}{2} \rfloor n = \frac{n(3m-1)}{4}$ monochromatic edges in the graph and the latter yields $\lceil \frac{m}{2} \rceil n + \lfloor \frac{m}{2} \rfloor \frac{n}{2} = \frac{n(3m+1)}{4}$ monochromatic edges. Thus, when the two $\delta^{(k)}$-colorings are compared the number of bad edges resulting from $\delta^{(2)}$-coloring of $P_m \circ C_n$ is $\frac{n(3m-1)}{4}$, when m is odd and n is even.

Case 3: Let m be even and n be odd. Since m is even, coloring the vertices of P_m alternatively with c_1 and c_2 or c_2 and c_1, will have same number of monochromatic edges in $P_m \circ C_n$. Thus, alternatively color the path P_m with the colors c_1 and c_2. The corresponding C_n's of each of the vertices in P_m that have received the color c_1 are alternatively assigned the color c_1 and c_2. This coloring will provide one monochromatic edge in each of these C_n and $\frac{m}{2}\lceil \frac{n}{2} \rceil$ of monochromatic edges between them. The remaining copies of C_n corresponding to $\frac{m}{2}$ vertices of P_m that have the color c_2, is colored with the color c_1 in order to maintain the requirements of a $\delta^{(k)}$-coloring of graphs. This coloring will cause for a situation where there are $\frac{nm}{2}$ monochromatic edges between them. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $P_m \circ C_n$ is $\frac{m}{2}\lceil \frac{n}{2} \rceil + \frac{nm}{2} + \frac{m}{2} = \frac{3m(n+1)}{4}$, when m is even and n is odd.

Case 4: Let m and n be odd. As explained in Theorem 4.5, there can be two possible $\delta^{(2)}$-colorings, one where the vertices of P_m are assigned the color c_1 and c_2 alternatively and the other vice versa. The former results in $\lceil \frac{m}{2} \rceil$ monochromatic edges in the cycles C_n that are given the colors c_1 and c_2 and $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ monochromatic edges between P_m and C_n. There are n monochromatic edges in $\lfloor \frac{m}{2} \rfloor$ cycle C_n that are corresponding to $\lfloor \frac{m}{2} \rfloor$ vertices of P_m whose color is c_2, which provides $n\lfloor \frac{m}{2} \rfloor$ monochromatic edges between these copies of C_n and P_m. Thus, the $\delta^{(2)}$-coloring is $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor n = \frac{3(m+mn+1)-n}{4}$ in $P_m \circ C_n$, when m and n are odd.

The latter will result in $\lfloor \frac{m}{2} \rfloor$ monochromatic edges in the $\lfloor \frac{m}{2} \rfloor$ copies of C_n which have the colors c_1 and c_2 and $\lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil$ monochromatic edges between P_m and C_n. Between $\lceil \frac{m}{2} \rceil$ vertices of P_m which have the color c_2 and its corresponding copies of C_n that are solely colored with c_1, there are $\lceil \frac{m}{2} \rceil n$ monochromatic edges. Thus, there are a total of $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil n = \frac{3(m+mn-1)+n}{4}$ monochromatic edges resulting from this coloring.

Now, when both the $\delta^{(2)}$-colorings are compared, the the number of bad edges resulting from $\delta^{(2)}$-coloring of $P_m \circ C_n$ is $\frac{3(m+mn+1)-n}{4}$, when both m and n are odd. □

Theorem 3.5. *For $P_m \circ K_n$, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is $b_k(P_m \circ K_n) = \frac{m(n-k+2)(n-k+1)}{2}$.*

Proof. It can be noted that, $\chi(P_m \circ K_n)$ is $n+1$ and hence $2 \leq k \leq n$. Color the vertices of P_n alternatively with the colors c_1 and c_2. There are $\lceil \frac{m}{2} \rceil$ vertices that receive the color c_1 and $\lfloor \frac{m}{2} \rfloor$ vertices that are given the color c_2. Each of the copies of K_n corresponding to each of the $\lceil \frac{m}{2} \rceil$ vertices of P_m that receive the color c_1 will cause $\lceil \frac{m}{2} \rceil \frac{(n-k+1)(n-k)}{2}$ monochromatic edges (see Theorem 2.7, [15], for the $\delta^{(k)}$-coloring of K_n) and $\lceil \frac{m}{2} \rceil (n-k+1)$ monochromatic edges between them. For the remaining $\lfloor \frac{m}{2} \rfloor$ copies of K_n's corresponding to the vertices that are assigned the color c_2 in P_m, there are $\frac{(n-k+2)(n-k+1)}{2}$ monochromatic edges. This is because, the color c_2 cannot be used to color K_n in order to maintain the conditions of a $\delta^{(k)}$-coloring of graphs. Also, there will not be any monochromatic edge between them. Thus, the total number of monochromatic edges obtained from this $\delta^{(k)}$-coloring is $\lceil \frac{m}{2} \rceil \frac{(n-k+1)(n-k)}{2} + \lceil \frac{m}{2} \rceil (n-k+1) + \lfloor \frac{m}{2} \rfloor \frac{(n-k+2)(n-k+1)}{2} = (n-k+1)(\lceil \frac{m}{2} \rceil \frac{(n-k)}{2} + \lfloor \frac{m}{2} \rfloor \frac{(n-k+2)}{2} + 1)$. In other words, it can be said that, in $P_m \circ K_n$ each vertex of P_m is adjacent to every vertex of K_n and hence there are m number of disjoint K_{n+1}. It is known that, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of K_{n+1} is $\frac{(n-k+1)(n-k)}{2}$ (see Theorem 2.7, [15]). Thus, in this case, each K_{n+1} will have $\frac{(n-k+2)(n-k+1)}{2}$ monochromatic edges. Thus, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $P_m \circ K_n$ is $\frac{m(n-k+2)(n-k+1)}{2}$. □

Theorem 3.6. *For $C_m \circ P_n$, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is*

$$b_2(C_m \circ P_n) = \begin{cases} \frac{3m(n-1)}{4}, & \text{if } m \text{ is even and for any } n, \\ \frac{(3m-1)(n-1)+4}{4}, & \text{if } m \text{ is odd and for any } n. \end{cases}$$

Proof. It is known that, $\chi(C_m \circ P_n) = 3$ and hence $k = 2$. The following are the two cases discussed for a $\delta^{(2)}$-coloring of $C_m \circ P_n$ for different parities of m and n.

Case 1: Let m be even. The cycle C_m of even length can be properly colored with two colors with $\frac{m}{2}$ possibility for each color, leading to no monochromatic edge in it. The $\frac{m}{2}$ copies of P_n, corresponding to $\frac{m}{2}$ vertices of C_m which have the color c_1, can be alternatively assigned the color c_2 and c_1 respectively, leading to a total of $\frac{m}{2}\lfloor\frac{n}{2}\rfloor$ monochromatic edges between them (Note that, if the P_ns are alternatively colored with the colors c_1 and c_2, there will be $\lceil\frac{n}{2}\rceil$ vertices that receive the color c_1, leading to $\lceil\frac{n}{2}\rceil$ monochromatic edges between the C_m and P_n which is more in number when compared to $\frac{m}{2}\lfloor\frac{n}{2}\rfloor$ monochromatic edges). The remaining $\frac{m}{2}$ copies of P_n are exclusively colored with c_1 as they are adjacent to the vertices of C_m which have the color c_2, to maintain the requirements of a $\delta^{(k)}$-coloring of graphs. This coloring will provide a situation where there are $\frac{m}{2}(n-1)$ monochromatic edges. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring is of $C_m \circ P_n$ is $\frac{m}{2}\lfloor\frac{n}{2}\rfloor + \frac{m}{2}(n-1) = \frac{3m(n-1)}{4}$, when m is even.

Case 2: Consider m to be odd. It can be noted that, the number of bad edges resulting from $\delta^{(2)}$-coloring of a cycle of odd length is 1, with $\lceil\frac{m}{2}\rceil$ vertices receiving the color c_1 and $\lfloor\frac{m}{2}\rfloor$ vertices the color c_2. As explained in the above-mentioned case, P_n's that are adjacent to its corresponding vertices that are assigned the color c_1 will yield a total of $\lceil\frac{m}{2}\rceil\lfloor\frac{n}{2}\rfloor$ monochromatic edges and P_n's adjacent to the vertices that have the color c_2 will lead in $\lfloor\frac{m}{2}\rfloor(n-1)$ monochromatic edges. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $C_m \circ P_n$ is $1+\lceil\frac{m}{2}\rceil\lfloor\frac{n}{2}\rfloor+\lfloor\frac{m}{2}\rfloor(n-1) = \frac{(3m-1)(n-1)+4}{4}$, when m is odd. □

Theorem 3.7. *For $C_m \circ K_n$, the $\delta^{(k)}$-coloring is*

$$b_k(C_m \circ K_n) \leq \begin{cases} \frac{m(n-k+1)(n-k+2)}{2}, & \text{if } m \text{ is even,} \\ \frac{m(n-k+1)(n-k+2)+2}{2}, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. The chromatic number of $C_m \circ K_m$ is $n+1$ and hence $2 \leq k \leq n$. For the different parities of m, there are two cases that are addressed separately as follows.

Case 1: Let m be even. It is known that, $\chi(C_{2n}) = 2$ and hence for any k, the even cycle C_m will yield no monochromatic edges. As explained in Theorem 3.6, every $\frac{m}{2}$ copies of K_n, adjacent to $\frac{m}{2}$ vertices of C_m, receiving the color c_1, will yield a total $\frac{m}{2}\frac{(n-k+1)(n-k)}{2}$ monochromatic edges in these K_n. Also, there are $\frac{m}{2}(n-k+1)$ monochromatic edges between these K_n and C_m. The remaining $\frac{m}{2}$ copies of K_n, adjacent to $\frac{m}{2}$ vertices of C_m, having the color other than c_1, cannot be assigned that particular color to meet the requirements of a $\delta^{(k)}$-coloring of graphs. Thus, these $\frac{m}{2}$ copies of K_n are colored with $k-1$ colors, leading to a total of $\frac{m}{2}\frac{(n-k+2)(n-k+1)}{2}$ monochromatic edges. Hence, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $C_m \circ K_n$ is $\frac{m}{2}\frac{(n-k+1)(n-k)}{2} + \frac{m}{2}(n-k+1) + \frac{m}{2}\frac{(n-k+2)(n-k+1)}{2} = \frac{m(n-k+1)(n-k+2)}{2}$, when m is even.

Case 2: Let m be odd. The minimum colors required to color an odd cycle is 3 and hence $2 \leq k \leq n$. When $k \geq 3$, C_m will cause to a scenario where there are no monochromatic edges. However, when $k = 2$, there will be a monochromatic edge in C_m. A common $\delta^{(k)}$-coloring for both the cases is discussed as follows. Color C_m with only two colors, say c_1 and c_2. This will cause a situation where there exists one monochromatic edge in C_m (see Proposition 2.3, [15]). As explained in Theorem 4.9 and Case 1 of the current theorem, the $\lceil\frac{m}{2}\rceil$ copies of K_n that are adjacent to $\lceil\frac{m}{2}\rceil$ vertices of C_m which have the color c_1, will result in $\lceil\frac{m}{2}\rceil\frac{(n-k+1)(n-k)}{2}$ and $\lfloor\frac{m}{2}\rfloor$ copies of K_n, adjacent to $\lfloor\frac{m}{2}\rfloor$ vertices of C_m which are assigned the color c_2, will have $\lfloor\frac{m}{2}\rfloor\frac{(n-k+2)(n-k-1)}{2}$ monochromatic edges. Now, between the vertices of C_m that receive the color c_1 and its corresponding copies of K_n, there are $\lceil\frac{m}{2}\rceil(n-k+1)$ monochromatic edges. Thus, the total number of monochromatic edges resulting from $\delta^{(k)}$-coloring of $C_m \circ K_n$ is $1 + \lceil\frac{m}{2}\rceil\frac{(n-k+1)(n-k)}{2} + \lfloor\frac{m}{2}\rfloor\frac{(n-k+2)(n-k-1)}{2} + \lceil\frac{m}{2}\rceil(n-k+1) = \frac{m(n-k+1)(n-k+2)+2}{2}$, when m is odd.

When $k \geq 3$, the odd cycle can properly be colored with three colors and this coloring will provide no monochromatic edge in C_m. However, there will be a total of $\lceil\frac{m}{2}\rceil$ vertices that receive the color other than the color c_1 and $\lfloor\frac{m}{2}\rfloor$ vertices that receive the color c_1, which will cause a situation where there exists $\lceil\frac{m}{2}\rceil\frac{(n-k+2)(n-k+1)}{2}$ and $\lfloor\frac{m}{2}\rfloor\frac{(n-k+1)(n-k)}{2}$ monochromatic edges between C_m and K_n. Now, when both the colorings are compared, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of above mentioned $\delta^{(k)}$-colorings are the same. This completes the proof. □

Theorem 3.8. *For $K_m \circ P_n$, the $\delta^{(k)}$-coloring is*

$$b_k(K_m \circ P_n) \leq \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 3, \\ \frac{(m-1)(m-2)}{2} + (n-1) + (m-1)\lfloor\frac{n}{2}\rfloor, & \text{if } k = 2. \end{cases}$$

Proof. The chromatic number of $K_m \circ P_n$ is m. Thus, the available colors are $2 \leq k \leq m-1$. There are two cases for two different values of k which are as explained below.

Case 1: Consider the case where $k \geq 3$. It is known that, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of K_n is $\frac{(n-k+1)(n-k)}{2}$ (see Theorem 2.7, [15]). Since the graph $K_m \circ P_n$ has a clique of order m, the minimum number of monochromatic edges that $b_k(K_m \circ P_n) \geq b_k(K_m)$. Is it proved that, in this case it is exactly $b_k(K_m)$. The minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of K_m is $\frac{(m-k+1)(m-k)}{2}$. Since $k \geq 3$, P_n can be properly colored with any two colors other than the color assigned to its corresponding vertex of K_m. Thus, there are no monochromatic edges between K_m and the m copies of P_n. Hence, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $K_m \circ P_n$ is $\frac{(m-k+1)(m-k)}{2}$.

Case 2: Let $k = 2$. Coloring the complete graph K_m with two colors will result in $\frac{(m-k+1)(m-k)}{2} = \frac{(m-1)(m-2)}{2}$ monochromatic edges. This is because, only one vertex say the vertex v_1 can be assigned the color c_2 and all the remaining vertices must be assigned with color c_1, to maintain the conditions of a $\delta^{(k)}$-coloring of graphs. Among the m copies of P_n the one which is adjacent to the vertex v_1 of K_m is colored with the color c_1, to meet the requirements of a $\delta^{(k)}$-coloring of graphs. This coloring will result in a situation where there are $n-1$ monochromatic edges in that particular P_n. The remaining $m-1$ copies of P_n, adjacent to the its corresponding vertices of K_m which have the color c_1, can be alternatively colored with the colors c_2 and c_1 respectively (and not c_1 and c_2 respectively, as it will maximise the use of the color c_1 and thereby maximise the number of monochromatic edges between them). Thus, this coloring will cause for a situation where there are $(m-1)\lfloor \frac{n}{2} \rfloor$ monochromatic edges between them. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $K_m \circ P_n$ is $\frac{(m-1)(m-2)}{2} + n - 1 + (m-1)\lfloor \frac{n}{2} \rfloor$. □

Theorem 3.9. *For $K_m \circ C_n$, when n is even, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is,*

$$b_k(K_m \circ C_n) = \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 3, \\ \frac{m(m+n-3)+n+2}{2}, & \text{if } k = 2. \end{cases}$$

Proof. For different values of k and when n is even, there are two different cases for a $\delta^{(k)}$-coloring of $K_m \circ C_n$. Since $\chi(K_m \circ P_n) = m$, $2 \leq k \leq m-1$. Considering all the above mentioned facts, both the cases are separately addressed as follows.

Case 1: Let $k \geq 3$. The proof explained in Case 1 of Theorem 3.8 applies to this case as well, this is because, paths and even cycles are bipartite and can be properly colored with two colors by maintaining the constraints of $\delta^{(k)}$-coloring, when $k \geq 3$. Thus, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $K_m \circ P_n$ is $\frac{(m-k+1)(m-k)}{2}$.

Case 2: Let $k = 2$. The proof of this case is similar to that of Case 2 of Theorem 3.8. The complete graph K_m will provide $\frac{(m-2)(m-1)}{2}$ monochromatic edges. The only difference is that, the cycle C_n which is adjacent to the vertex (only vertex) that is assigned the color c_2 is given the color c_1, which yields n monochromatic edges in the cycle. All the remaining $m-1$ copies of C_n are assigned the color c_1 and c_2 alternatively, which results in $(m-1)\frac{n}{2}$ monochromatic edges between K_m and $(m-1)$ copies of C_n. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $K_m \circ C_n$ is $\frac{(m-2)(m-1)}{2} + (m-1)\frac{n}{2} + n = \frac{m(m+n-3)+n+2}{2}$, when n is even. □

Theorem 3.10. *For $K_m \circ C_n$, when n is odd, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is,*

$$b_k(K_m \circ C_n) \leq \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 4, \\ \frac{(m-2)(m-1)+4}{2}, & \text{if } k = 3, \\ \frac{m(m+n)+n-1}{2}, & \text{if } k = 2. \end{cases}$$

Proof. Note that, $\chi(K_m \circ C_n) = m$ and hence the $\delta^{(k)}$-coloring of the same for the different values of k, where $2 \leq k \leq m-1$, are studied. There are three different cases for the same that are to be addressed separately as follows.

Case 1: Let $k \geq 4$. The minimum number of monochromatic edges obtained from $\delta^{(k)}$-coloring of $K_m \circ C_n$ is $\frac{(m-k+1)(m-k)}{2}$, when $k \geq 4$. The proof of this case is same as that of the proof explained in Case 1 of Theorem 3.8 and Theorem 3.9.

Case 2: Let $k = 3$. The complete graph K_m will yield $\frac{(m-k+1)(m-k)}{2} = \frac{(m-2)(m-3)}{2}$ monochromatic edges, when colored with three colors (see Theorem 2.7, [15]). There are only two vertices in K_m, say v_1 and v_2, that can be colored with the colors c_2 and c_3. Rest of the vertices have to be given the color c_1, to meet the requirements of a $\delta^{(k)}$-coloring of graphs. Since C_n is an odd cycle, it will require at least three colors to color it properly. Although, the number of available colors is 3 these colors are used in the coloring of K_m and hence each $n-2$ copies of cycle corresponding to $n-k$ vertices of K_n that have the color c_1 are colored with two colors c_2 and c_3 and this coloring will cause a minimum of one monochromatic edges in the cycle and between K_m and its corresponding C_n. Moreover, the vertex v_1 of K_m is assigned the color c_2 and hence the cycle corresponding to this vertex is colored with two colors c_1 and c_3 leading to no monochromatic edge between them. However, there will be a monochromatic edge in C_n when colored with two colors (see Proposition 2.3, [15]). Similarly, in the case of the vertex v_2 that is assigned the color c_3, its corresponding C_n will cause one monochromatic edge when colored with the colors c_1 and c_2. Thus, the number of bad edges resulting from $\delta^{(3)}$-coloring of $K_m \circ C_n$ is $\frac{(m-2)(m-3)}{2} + 2 + (m-2) = \frac{(m-2)(m-1)+4}{2}$, when n is odd.

Case 3: Let $k = 2$. As explained in Case 2 of Theorem 3.9, only one vertex, say v_1, of K_m is given the color c_2 and the rest of the vertices are colored with the color c_1. This coloring will cause a situation where there are $\frac{(m-1)(m-2)}{2}$ monochromatic edges. Now, C_n corresponding to the vertex v_1 is solely colored with c_1 to meet the requirements of a $\delta^{(k)}$-coloring of graphs, and this coloring causes n monochromatic edges in this particular cycle. The remaining copies of C_n are colored with two colors c_1 and c_2, leading to one monochromatic edge in each of the $m-1$ copies of C_n and $(m-1)\lceil \frac{n}{2} \rceil$ monochromatic edges between K_m and C_n. Thus, the $\delta^{(2)}$-coloring of $K_m \circ C_n$ is $\frac{(m-1)(m-2)}{2} + (m-1) + (m-1)\lceil \frac{n}{2} \rceil + n = \frac{m(m+n)+n-1}{2}$, when n is odd, as required □

Theorem 3.11. *For $K_m \circ K_n$, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is given by,*

$$b_k(K_m \circ K_n) = \begin{cases} (m-k+1)\left(\frac{m-k}{2} + n - k + 1\left(\frac{n-k+2}{2}\left(\frac{m}{m-k+1}\right)\right)\right), & \text{if } k \geq 3, \\ \frac{(n(n-3)+m)(m-1)+2n}{2}, & \text{if } k = 2. \end{cases}$$

Proof. Each of the copies of K_n corresponding to the each of the vertex assigned the color c_1 in K_m will lead in $\frac{(n-k+1)(n-k)}{2}$ monochromatic edges and between them there will be $(m-k+1)(n-k+1)$ monochromatic edges (for a detailed explanation on the coloring pattern of $\delta^{(k)}$-coloring of complete graphs see Theorem 2.7, [15]). Now, $k-1$ copies of K_n corresponding to $k-1$ vertices that receive the color other than c_1 in K_m can be colored with $k-1$ colors only (the color assigned to its corresponding vertex in K_m, cannot be used in coloring its corresponding K_n). This coloring will provide a situation where there are $(k-1)\frac{(n-k+1)(n-k)}{2}$ monochromatic edges between them. Thus, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $K_m \circ K_n$ is $\frac{(m-k+1)(m-k)}{2} + (m-k+1)\frac{(n-k+1)(n-k)}{2} + (m-k+1)(n-k+1) + (k-1)\frac{(n-k+1)(n-k)}{2} = (m-k+1)(\frac{m-k}{2} + n - k + 1(\frac{n-k+2}{2}(\frac{m}{m-k+1})))$, when $k \geq 3$.

Case 2: Let $k = 2$. Coloring K_m with two colors will result in a scenario where there are $\frac{(m-k+1)(m-k)}{2} = \frac{(m-1)(m-2)}{2}$ monochromatic edges. All the corresponding copies of K_n, other than the one which is adjacent to the vertex assigned the color c_2 of K_m, are colored with two colors, leading to $\frac{(m-k+1)(n-k+1)(n-k)}{2} = \frac{(m-1)(n-1)(n-2)}{2}$ monochromatic edges in the K_m. Between the vertices of K_m having the color c_1, that is, $m-1$ vertices of K_m and $m-1$ copies of K_n there are $(m-k+1)(n-k+1) = (m-1)(n-1)$ monochromatic edges. The complete graph K_n adjacent to the vertex colored with the color c_2 of K_m should be given only the color c_1 to maintain the requirements of a $\delta^{(k)}$-coloring of graphs, leading to $\frac{n(n-1)}{2}$ monochromatic edges. Thus, the number of bad edges resulting from $\delta^{(2)}$-coloring of $K_m \circ K_n$ $\frac{(m-1)(m-2)}{2} + \frac{(m-1)(n-1)(n-2)}{2} + (m-1)(n-1) + \frac{n(n-1)}{2} = \frac{(n(n-3)+m)(m-1)+2n}{2}$. □

4 $\delta^{(k)}$-coloring of graph products

Recall that the *direct product* of G and H is the graph denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$, and for which the vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. Thus,

(i) $V(G \times H)=\{(g, h) \mid g \in V(G) \text{ and } h \in V(H)\}$,

(ii) $E(G \times H)=\{(g, h)(g', h') \mid gg' \in E(G) \text{ and } hh' \in E(G)\}$ (see [12, 10]).

In a direct product of two graphs with m and n vertices respectively, there are a total of $m \times n$ vertices. Thus, there are m set each of n vertices or vice versa in the direct product. Throughout the discussion, we consider $n \geq m$ (note that, since direct product is a commutative product, all the results discussed below hold for $n < m$ as well) and that there are m set each of n vertices. The first set of n vertices is denoted as g_1h_j where $1 \leq j \leq n$, the second set is denoted as g_2h_j where $1 \leq j \leq n$ and so on the m-th set of n vertices is denoted as g_mh_j where $1 \leq j \leq n$.

Other names for the direct product that appears in the literature are *tensor product*, *Kronecker product*, *conjunction*, *cross product* etc. If either of a graph G or H in the direct product is bipartite then their direct product $G \times H$ is bipartite and hence the following discussion does not consider the $\delta^{(k)}$-coloring of path graph and or even cycle and their various combinations. This paper solely focuses on cycle graph C_n for odd n and complete graph

K_n. Now, $\chi(G \times H)$ is less than or equal to $\min(\chi(G), \chi(H))$ and hence when $C_m \times C_n$ and $C_m \times K_n$ are considered, $\chi(C_m \times C_n) = 3$ when both n and m are odd and $\chi(C_m \times K_n) = 3$, when m is odd and hence for these two cases, a $\delta^{(2)}$-coloring of the same is considered. For $K_m \times K_n$, the value of k will be $2 \leq k \leq \min(m, n) - 1$. The direct product is commutative and hence the concerned study focuses on either of the $\delta^{(k)}$-coloring of $G \times H$ or $H \times G$. The following are the results obtained from a $\delta^{(k)}$-coloring of direct product of cycle graph and complete graph with their possible combination.

Theorem 4.1. *For $C_m \times C_n$ where m and n are odd and $m \leq n$, the minimum number of bad edges obtained from $\delta^{(2)}$-coloring is given by*

$$b_2(C_m \times C_n) = 2m.$$

Proof. For $C_m \times C_n$ where $n \geq m$, the chromatic number is 3 and it is a 4 regular graph. Hence, in this case the value of k can only be 2. Now, it is clear from the definition of $\delta^{(k)}$-coloring that every color class other than C_1 is an independent set. As we determine the minimum number of bad edges resulting from $\delta^{(k)}$-coloring, it is clear that a $\delta^{(2)}$-coloring of a graph is based on the independence number of the graph. The independence number of $G \times H$ is given as $\alpha(G \times H) \geq \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$ (see [10]). Now, the independence number of direct product of odd cycles is already been discussed in [14] as $(n-1)\frac{m}{2}$. Now, the minimum number of bad edges obtained from $\delta^{(k)}$-coloring for an r-regular graph, with n vertices when $k = 2$ and α is the independence number is discussed in [6], is $\frac{r(n-2\alpha)}{2}$. Thus, in this case, the minimum number of bad edges obtained from $\delta^{(2)}$-coloring of $C_m \times C_n$ is given as $b_2(C_m \times C_n) = \frac{4(mn-2(n-1)\frac{m}{2})}{2} = 2m$. □

Theorem 4.2. *For $C_m \times K_n$ where m and n are odd, the minimum number of bad edges obtained from $\delta^{(2)}$-coloring is given by*

$$b_2(C_m \times K_n) = n(n-1).$$

Proof. We know that, $\chi(C_m \times K_n) = 3$ and hence the only value that k can take in this case is 2. As explained in Theorem 4.1, we first use the concept of independence number and provide an upper bound for the minimum number of bad edges obtained from $\delta^{(2)}$-coloring. We know that, the independence number of $G \times H$ is given as

$$\alpha(G \times H) \geq \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$$

(see [10]). Thus, in this case we have,

$$\alpha(C_m \times K_n) \geq \max\{\alpha(C_m)|V(K_n)|, \alpha(K_n)|V(C_m)|\} = \max\{n\lfloor\frac{m}{2}\rfloor, m\}.$$

Since we consider $m \leq n$, here we have,

$$\alpha(C_m \times K_n) \geq n\lfloor \frac{m}{2} \rfloor.$$

Now, the minimum number of bad edges resulting from $\delta^{(2)}$-coloring of a regular graph is $\frac{r(n-2\alpha)}{2}$ (see [6]). Thus, we have,

$$b_2(C_m \times K_n) \geq \frac{r(n-2\alpha)}{2} \geq \frac{2(n-1)(mn - 2n\lfloor \frac{m}{2} \rfloor)}{2} \geq n(n-1).$$

Hence, the minimum number of bad edges resulting from a $\delta^{(2)}$-coloring of $C_m \times K_n$ is, $b_2(C_m \times K_n) \geq n(n-1)$.

Now, we prove that $b_2(C_m \times K_n)$ is exactly equal to $n(n-1)$ by providing a $\delta^{(2)}$-coloring that results in the same. Here, none of the g_1h_j where $1 \leq j \leq n$ are adjacent to each other, all the g_1h_j can be assigned a single color say c_1. Now, each g_1h_j is adjacent to every g_2h_j except for its corresponding vertex. Hence, the vertices g_2h_j can be assigned the color c_2 or c_1. However, our aim is to minimise the number of bad edges and so we use the color c_2 to color g_2h_j. The next n set of vertices g_3h_j can be colored with the color c_1 and the other set g_4h_j can be assigned the color c_2. Thus, we can alternatively color each n set with two colors c_1 and c_2 properly. Now, the last set of n vertices, g_mh_j where $1 \leq j \leq n$, has to be assigned the color c_1 to maintain the definition of $\delta^{(k)}$-coloring. Now, the only edges that lead to bad edges are the edges between the first set of vertices (g_1h_j) and the m-th set of vertices (g_mh_j). Each of the n vertices in the set g_1h_j that are assigned the color c_1 are adjacent to $n-1$ vertices of the set g_mh_j given the color c_1, leading to a total of $n(n-1)$ bad edges. Thus, the minimum number of bad edges between the $C_m \times K_n$ resulting from $\delta^{(2)}$-coloring is $n(n-1)$. □

Theorem 4.3. *For $K_m \times K_n$ where m and n are odd and $n \geq m$, the minimum number of bad edges obtained from $\delta^{(k)}$-coloring is given by*

$$b_k(K_m \times K_n) = \frac{n(n-1)(m-k)(m-k+1)}{2}.$$

Proof. The chromatic number, $\chi(K_m \times K_n) = \min\{m, n\}$. Now, since we consider $n \geq m$, $\chi(K_m \times K_n) = m$ and hence k can be $2 \leq k \leq m-1$. In this case, there can be two possible $\delta^{(k)}$-colorings which are as explained below. In $K_m \times K_n$, every vertex is adjacent to every other vertex except its corresponding vertices. Thus, either every corresponding vertex, which is an independent set, can be assigned a single color or every n vertices in a single set, which is an independent set, can be given a single color. Now, since $2 \leq k \leq m-1$, in the former case, $n-k+1$ independent set of m vertices will receive the color c_1 and each m vertices assigned the color c_1 is adjacent to $m-1$ vertices assigned the color c_1. Similarly, in the later case $m-k+1$ independent set of n vertices will receive the color c_1 each n vertices assigned the color c_1 is adjacent to $n-1$ vertices assigned the color c_1. Since there are $n-k+1$ and $m-k+1$ independent set colored only with the single color c_1 and since every vertex is adjacent to every other vertex other than its corresponding vertex, both a $\delta^{(k)}$-colorings will lead to $\frac{m(m-1)(n-k)(n-k+1)}{2}$ and $\frac{n(n-1)(m-k)(m-k+1)}{2}$ bad edges respectively. Now, since $n \geq m$, the minimum number of bad edges obtained when both the $\delta^{(k)}$-colorings are compared is, $\frac{n(n-1)(m-k)(m-k+1)}{2}$ when $n > m$ and both are same when $n = m$. Thus, the minimum number of bad edges obtained from $\delta^{(k)}$-coloring of $K_m \times K_n$ when $n \geq m$, $\frac{n(n-1)(m-k)(m-k+1)}{2}$. □

Theorem 4.4. *For any graph G and H, the minimum number of bad edges obtained from $\delta^{(k)}$-coloring of direct product $G \times H$, is given by,*

$$b_k(G \times H) \leq \frac{n(n-1)(m-k)(m-k+1)}{2}.$$

Proof. Since the maximum number of edges on m and n vertices is the complete graph K_m and K_n respectively, it is clear that, the maximum number of edges a direct product of two graph G and H can have is $|E(K_m \times K_n)|$. Now, it can be noted that, any direct product $G \times H$ is a subgraph of $K_m \times K_n$. Thus, it can be concluded that, $b_k(G \times H) \leq b_k(K_m \times K_n)$. □

The corona product of G and H is the graph $G \circ H$ obtained by taking one copy of G, called the centre graph, $|V(G)|$ copies of H, called the outer graph, and making the i-th vertex of G adjacent to every vertex of the i-th copy of H, where $1 \leq i \leq |V(G)|$ (see [8]). The corona product is not commutative and hence all the possible combination of $\delta^{(k)}$-coloring of corona product of path graph, cycle graph and complete graphs are taken into consideration in this paper.

Theorem 4.5. *For $P_m \circ P_n$, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is given by*

$$b_2(P_m \circ P_n) = \min\left\{\left\lceil\frac{m}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{m}{2}\right\rfloor(n-1), \left\lceil\frac{m}{2}\right\rceil(n-1) + \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\right\},$$

for any m and n.

Proof. The graph $P_m \circ P_n$ is 3-colorable and hence k can take only the value 2. There can be two possible $\delta^{(k)}$-coloring in this case as explained below. The first coloring is to color the path graph P_m with the vertices $v_1, v_2, \ldots, v_m$ with two colors c_1 and c_2 alternatively. This coloring will lead to $\lceil \frac{m}{2} \rceil$ independent vertices that are assigned the color c_1 and $\lfloor \frac{m}{2} \rfloor$ independent vertices with the color c_2. Now, the P_ns corresponding to the vertices in P_m that are assigned the color c_1, can be alternatively assigned the color c_1 and c_2. Again, similar to the coloring of P_m, if we start coloring the P_n with color c_1, there will be $\lceil \frac{n}{2} \rceil$ independent vertices assigned the color c_1 and $\lfloor \frac{n}{2} \rfloor$ independent vertices that are assigned the color c_2 and vice versa when started coloring it with the color c_2. The former will increase the number of bad edges due to the increase in the number of vertices that receive the color c_1 and the later will decrease the same by 1. Hence, we start coloring the P_n with the color c_2 and then assign the next vertex the color c_1 and so on. Now, there are $\lceil \frac{m}{2} \rceil$ vertices in P_m colored with color c_1, adjacent to the vertices of its corresponding path graph P_n, that have $\lfloor \frac{n}{2} \rfloor$ vertices assigned the color c_1. This lead to $\lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor$ bad edges between them. Now, the P_ns, corresponding to the $\lfloor \frac{m}{2} \rfloor$ vertices in P_m that receive the color c_2, should be assigned the color c_1 to maintain the definition of $\delta^{(k)}$-coloring. Thus, there will not be any bad edge between P_m and P_n in this case. However, every edge in P_n will be a bad edge, leading to a total of $\lfloor \frac{m}{2} \rfloor (n-1)$ bad edges. Thus, the total number of bad edges resulting from this particular $\delta^{(k)}$-coloring is $\lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor (n-1)$.

In the second $\delta^{(k)}$-coloring, start coloring the vertices of P_m alternatively with the colors c_2 and c_1. This will lead to $\lceil \frac{m}{2} \rceil$ independent vertices that are assigned the color c_2 and $\lfloor \frac{m}{2} \rfloor$ independent vertices with the color c_1, which thereby leads in coloring $\lceil \frac{n}{2} \rceil$ P_n's solely with the color c_1, leading to $\lceil \frac{m}{2} \rceil (n-1)$ bad edges. Now, the remaining vertices of the corresponding P_ns, of the $\lfloor \frac{m}{2} \rfloor$ vertices of P_m that are assigned the color c_1, are assigned the color c_2 and c_1 alternatively leading to $\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$ bad edges. Thus, the total number of bad edges resulting from this $\delta^{(k)}$-coloring $\lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$. Now, when both the colorings are compared, the bad edges obtained from both is the same when m is even and is the $\min\{\lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor (n-1), \lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor\}$ when m is odd. □

Theorem 4.6. *For $C_m \circ C_n$, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is given by*

$$b_k(C_m \circ C_n) = \begin{cases} \min\{\frac{n(3m-1)+4}{4}, \frac{m(n+2)}{2}\}, & \text{if } m \text{ is odd, } n \text{ is even and } k=2, \\ \min\{\frac{3n(n+1)}{4}, \frac{m(n+3)+2n}{2}\}, & \text{if } m \text{ is even, } n \text{ is odd and } k=2, \\ m, & \text{if } m,\ n \text{ are both odd and } k=3, \\ \min\{\frac{3mn}{4}, \frac{m(n+2)}{2}\}, & \text{if } m,\ n \text{ are both even and } k=2. \end{cases}$$

Proof. The different cases for a $\delta^{(k)}$-coloring of $C_m \circ C_n$, for different parities of m and n and for different values of k are explained as below.

Case 1: Let m be odd and n be even. The odd cycle C_m when colored with two colors c_1 and c_2 will lead to 1 bad edges (see [15]). Now, the C_n where n is even can be properly colored using two colors. However, since the $\lfloor \frac{m}{2} \rfloor$ vertices of C_m are assigned the color c_2, its corresponding C_n's ($\lfloor \frac{m}{2} \rfloor$ C_n's) should be colored only with c_1 to meet the requirements of $\delta^{(k)}$-coloring. This will lead to $\lfloor \frac{m}{2} \rfloor n$ bad edges. Also, the $\lceil \frac{m}{2} \rceil$ vertices of C_m that are assigned the color c_1 adjacent to its corresponding C_n's are assigned the color c_1 and c_2, will lead to $\lceil \frac{m}{2} \rceil \frac{n}{2}$ bad edges. Thus, the total number of bad edges obtained from $\delta^{(k)}$-coloring of $C_m \circ C_n$ when m is odd and n is even $\lfloor \frac{m}{2} \rfloor n + \lceil \frac{m}{2} \rceil \frac{n}{2} + 1 = \frac{n(3m-1)+4}{4}$. Now, another possible $\delta^{(k)}$-coloring for this case is that, the cycle C_m is colored with a single color c_1, leading to m bad edges and the m C_n's are alternatively assigned the color c_1 and c_2. This will lead to $\frac{mn}{2}$ bad edges between C_m and C_n. Thus, the total number of bad edges obtained from this $\delta^{(k)}$-coloring is $m + \frac{mn}{2} = \frac{m(n+2)}{2}$. Hence, the minimum total number of bad edges obtained from $\delta^{(k)}$-coloring of $C_m \circ C_n$ when m is odd and n is even is $\min\{\frac{n(3m-1)+4}{4}, \frac{m(n+2)}{2}\}$.

Case 2: Let m be even and n be odd. Since m is even, coloring C_m with two colors c_1 and c_2 will lead to no bad edges in C_m. Now, the $\lceil \frac{n}{2} \rceil$ vertices of C_n are assigned the color c_1 and $\lfloor \frac{n}{2} \rfloor$ vertices are assigned the color c_2, leading to one bad edge in each of $\frac{m}{2}$ C_n's and $\frac{m}{2} \lfloor \frac{n}{2} \rfloor$ between them. The remaining C_n's that are adjacent to the vertices of C_m ($\lfloor \frac{m}{2} \rfloor$ vertices) which are assigned the color c_2, are solely colored with the color c_1, to meet the definition of $\delta^{(k)}$-coloring. This will lead to get $\lfloor \frac{m}{2} \rfloor n$ bad edges. Thus, the total number of bad edges resulting from this $\delta^{(k)}$-coloring is $\frac{m}{2} \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor n + \frac{m}{2} = \frac{3n}{2}(\frac{n+1}{2})$. Now, the second possible $\delta^{(k)}$-coloring for this case is same as that of second $\delta^{(k)}$-coloring explained in Case 1. This coloring will lead to all the edges in C_m to be bad (since all the m vertices are assigned the color c_1). However, since n is odd in this case, there will be 1 bad edge in each of the n copies of C_n and $m \lceil \frac{n}{2} \rceil$ bad edges between C_n and C_m. Thus, the total number of bad edges in this case is $\frac{m(n+3)+2n}{2}$. Thus, the minimum number of bad edges resulting from a $\delta^{(k)}$-coloring of $C_m \circ C_n$ when m is even and n is odd is $\min\{\frac{3n(n+1)}{4}, \frac{m(n+3)+2n}{2}\}$.

Case 3: Let m and n be odd and $k = 3$. The chromatic number of $C_m \circ C_n$ when m and n are odd is 4 and hence, we have $k = 3$ and $k = 2$. First we discuss a $\delta^{(3)}$-coloring of $C_m \circ C_n$. Since $k = 3$, we maximise the use of the colors c_2 and c_3 and minimise the use of color c_1 as much as possible. A $\delta^{(k)}$-coloring that exactly explains the same is as follows. Assign the vertices of C_m alternatively with the colors c_2 and c_3 and the last vertex v_m is assigned the color c_1. This is a proper coloring of an odd cycle with $k = 3$ colors. Now, each of the $m-1$ C_n's corresponding to the vertices of the C_m assigned the color c_2 and c_3 can be given the colors c_1 and c_3, and c_1 and c_2 respectively. This will lead to 1 bad edge in each of the $m-1$ C_n's. Now, the corresponding C_n of the v_mth vertex of C_m that is assigned the color c_1 can be properly colored with $k = 3$ colors, leading to 1 bad edge between C_m and C_n. Thus, the total number of bad edges resulting from this $\delta^{(k)}$-coloring which minimises the use of color c_1 is m.

Case 4: Let m and n be odd and $k = 2$. In this case, we have two $\delta^{(k)}$-colorings as explained in the above cases. The first $\delta^{(k)}$-coloring is alternatively coloring odd C_m with $k = 2$ colors leading to 1 bad edge in C_m. Now, the C_ns, corresponding to the $\lceil \frac{m}{2} \rceil$ vertices of C_m that are colored with c_1 and $\lfloor \frac{m}{2} \rfloor$ vertices are assigned the color c_2, are assigned the color c_1 and c_2 alternatively. This will lead to have a total of $1 + \lceil \frac{m}{2} \rceil + n\lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil = \frac{3m(1+n)-n+7}{4}$ bad edges. The second $\delta^{(k)}$-coloring is same as the second $\delta^{(k)}$-coloring of Case 1. The C_m is exclusively colored with the color c_1 and the corresponding C_ns are assigned the color c_1 and c_2 (see [15] for a $\delta^{(k)}$-coloring of an odd cycle). This $\delta^{(k)}$-coloring will lead us to have $m+m+m\lceil \frac{n}{2} \rceil = \frac{m(n+5)}{2}$ bad edges. Thus, the $\min\{\frac{3m(1+n)-n+7}{4}, \frac{m(n+5)}{2}\}$ is the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $C_m \circ C_n$ when both m and n are odd and $k = 2$.

Case 5: Let m and n be even and $k = 2$. The $\chi(C_m \circ C_n)$ when both m and n are even is 3. Hence, the only value that k can take in this case is 2. We discuss two possible $\delta^{(k)}$-colorings for this case. In the first $\delta^{(k)}$-coloring, since m is even, the C_m can be properly colored with $k = 2$ colors. Now, each of the C_n's adjacent to the $\frac{m}{2}$ vertices of C_m assigned the color c_1 are given the color c_1 and c_2 alternatively leading to $\frac{mn}{4}$ bad edges between them. Now, the $\frac{m}{2}$ C_n's that are adjacent to $\frac{m}{2}$ vertices of C_m assigned the color c_2, are solely colored with c_1, to maintain the definition of $\delta^{(k)}$-coloring. Thus, this leads to getting $\frac{mn}{2}$ bad edges. Hence, the total number of minimum bad edges obtained from this $\delta^{(k)}$-coloring is $\frac{3mn}{4}$. Now, the second $\delta^{(k)}$-coloring is same as that of the second $\delta^{(k)}$-coloring explained in Case 1 and the number of bad edges obtained from this case is $\frac{m(n+2)}{2}$. Thus, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring of $C_m \circ C_n$ when both m and n are even and $k = 2$ is $\min\{\frac{3mn}{4}, \frac{m(n+2)}{2}\}$. □

Theorem 4.7. *For $P_m \circ C_n$, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is given by*

$$b_2(P_m \circ C_n) = \begin{cases} \frac{3mn}{4}, & \text{if } m \text{ and } n \text{ are even,} \\ \frac{3mn-n}{4}, & \text{if } m \text{ is odd and } n \text{ is even,} \\ \frac{3m(n+1)}{4}, & \text{if } m \text{ is even and } n \text{ is odd,} \\ \frac{3(m+nm+1)-n}{4}, & \text{if } m \text{ and } n \text{ are odd.} \end{cases}$$

Proof. The chromatic number of $P_m \circ C_n$ for any m and n is 3. Thus, the only value k can take in this case is 2. Now, for different parities of m and n, different $\delta^{(k)}$-colorings and the number of bad edges obtained from the same is as explained below.

Case 1: Let m and n be even. In this particular case, coloring P_m alternatively with c_1 and c_2 or vice versa will lead to same number of bad edges in the graph and hence we color the P_m alternatively with c_1 and c_2. This will lead to no bad edges in P_m. However, the vertices, of the corresponding C_n of the vertices that are assigned the color c_1 in P_m, are assigned the color c_1 and c_2 alternatively and the remaining C_n's are assigned exclusively assigned the color c_1 to maintain the requirements of $\delta^{(k)}$-coloring. Thus, there are a total of $\frac{m}{2}\frac{n}{2} + \frac{m}{2}n = \frac{3mn}{4}$ bad edges resulting from a $\delta^{(k)}$-coloring of $P_m \circ C_n$ when both m and n are even.

Case 2: Let m be odd and n is even. As explained in Theorem 4.5, there can be two possible $\delta^{(k)}$-colorings for this case. The first coloring is when the P_m is alternatively colored with the colors c_1 and c_2 and the second one the vertices of P_m is assigned the colors c_2 and c_1 alternatively. The former will create $\lceil \frac{m}{2} \rceil \frac{n}{2} + \lfloor \frac{m}{2} \rfloor n = \frac{n(3m-1)}{4}$ bad edges in the graph and the later creates $\lceil \frac{m}{2} \rceil n + \lfloor \frac{m}{2} \rfloor \frac{n}{2} = \frac{n(3m+1)}{4}$ bad edges. Thus, when the two $\delta^{(k)}$-colorings are compared the minimum bad edges obtained from a $\delta^{(k)}$-coloring of $P_m \circ C_n$ when m is odd and n is even is $\frac{n(3m-1)}{4}$.

Case 3: Let m be even and n be odd. Since m is even, coloring the vertices of P_m alternatively with c_1 and c_2 or c_2 and c_1, will lead to same number of bad edges in $P_m \circ C_n$. Thus, we alternatively color the P_m with the colors c_1 and c_2. Now, the corresponding C_ns of each of the vertices in P_m that have received the color c_1 are alternatively assigned the color c_1 and c_2. This will generate 1 bad edge in each of the C_n since n is odd and $\frac{m}{2}\lceil \frac{n}{2} \rceil$ of bad edges between them. Now, the $\frac{m}{2}$ vertices of P_m that receive the color c_2, its corresponding C_n is solely given the color c_1 to maintain the requirements of $\delta^{(k)}$-coloring. This will lead to $\frac{nm}{2}$ bad edges between them. Thus, the total number of bad edges in $P_m \circ C_n$ when m is even and n is odd is $\frac{m}{2}\lceil\frac{n}{2}\rceil + \frac{nm}{2} + \frac{m}{2} = \frac{3m(n+1)}{4}$.

Case 4: Let m and n be odd. As explained in Theorem 4.5, since m is odd there can be two possible $\delta^{(k)}$-colorings, one where the vertices of P_m are assigned the color c_1 and c_2 alternatively and the other vice versa. The former will lead to $\lceil\frac{m}{2}\rceil$ bad edges in the respective cycles C_n that are colored with two colors and $\lceil\frac{m}{2}\rceil\lceil\frac{n}{2}\rceil$ bad edges between P_m and C_n whose $\lceil\frac{m}{2}\rceil$ vertices are assigned the color c_1 and $\lceil\frac{n}{2}\rceil$ vertices are given the color c_1 respectively. Now, there are n bad edges in $\lfloor\frac{m}{2}\rfloor$ C_ns that are corresponding to $\lfloor\frac{m}{2}\rfloor$ vertices of P_m that are assigned the color c_2, leading to $\lfloor\frac{m}{2}\rfloor n$ bad edges. Thus, the total minimum number of bad edges resulting from this $\delta^{(k)}$-coloring when m and n are odd is $\lceil\frac{m}{2}\rceil\lceil\frac{n}{2}\rceil + \lceil\frac{m}{2}\rceil + \rfloor\frac{m}{2}\lfloor n = \frac{3(m+mn+1)-n}{4}$ in $P_m \circ C_n$. Now, the later will lead to $\lfloor\frac{m}{2}\rfloor$ bad edges in the respective cycles that are colored with two colors and $\lfloor\frac{m}{2}\rfloor\lceil\frac{n}{2}\rceil$ bad edges between P_m and C_n whose $\lfloor\frac{m}{2}\rfloor$ vertices are assigned the color c_1 and $\lceil\frac{n}{2}\rceil$ vertices are given the color c_1 respectively. Between the $\lceil\frac{m}{2}\rceil$ vertices of P_m that are assigned the color c_2 and its corresponding C_n's that are only colored with c_1, there are $\lceil\frac{m}{2}\rceil n$ bad edges. Thus, there are a total of $\lfloor\frac{m}{2}\rfloor + \lfloor\frac{m}{2}\rfloor\lceil\frac{n}{2}\rceil + \lceil\frac{m}{2}\rceil n = \frac{3(m+mn-1)+n}{4}$ bad edges resulting from this coloring. Now, when both the $\delta^{(k)}$-colorings are compared, the minimum number of bad edges resulting from the $\delta^{(k)}$-coloring of $P_m \circ C_n$ when both m and n are odd is $\frac{3(m+mn+1)-n}{4}$. □

Theorem 4.8. *For $P_m \circ K_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-coloring where $1 \leq k \leq \chi(P_m \circ K_n) - 1$ is given by*

$$b_k(P_m \circ K_n) = \frac{m(n-k+2)(n-k+1)}{2},$$

for all m and n.

Proof. The minimum number of colors required to color $P_m \circ K_n$ is $n+1$, hence, in this case the k can take the values from 2 to n. Now, color the vertices of the path graph alternatively with the colors c_1 and c_2. Now, there are $\lceil\frac{m}{2}\rceil$ vertices that receive the color c_1 and $\lfloor\frac{m}{2}\rfloor$ vertices that are assigned the color c_2 in P_m. Now, each of the K_n's corresponding to each of the $\lceil\frac{m}{2}\rceil$ vertices that receive the color c_1 will lead to $\lceil\frac{m}{2}\rceil\frac{(n-k+1)(n-k)}{2}$ bad edges (see [15], for the $\delta^{(k)}$-coloring of K_n) and $\lceil\frac{m}{2}\rceil(n-k+1)$ bad edges between them. For the remaining $\lfloor\frac{m}{2}\rfloor$ K_n's corresponding to the vertices that are assigned the color c_2 in P_m, there are $\frac{(n-k+2)(n-k+1)}{2}$ bad edges. This is because, the color c_2 cannot be used to color the K_n's in this case to maintain the definition of $\delta^{(k)}$-coloring. Also, there will not be any bad edge between them. Thus, the total number of bad edges obtained from this $\delta^{(k)}$-coloring for $P_m \circ K_n$ for any m and n is $\lceil \frac{m}{2} \rceil \frac{(n-k+1)(n-k)}{2} + \lceil \frac{m}{2} \rceil (n-k+1) + \lfloor \frac{m}{2} \rfloor \frac{(n-k+2)(n-k+1)}{2} = (n-k+1)(\lceil \frac{m}{2} \rceil \frac{(n-k)}{2} + \lfloor \frac{m}{2} \rfloor \frac{(n-k+2)}{2} + 1)$. In other words, we can say that, in $P_m \circ K_n$ each vertex of P_m is adjacent to every vertex of K_n and hence there are m number of disjoint K_{n+1}. We know that the minimum number of bad edges obtained from K_n is $\frac{(n-k+1)(n-k)}{2}$ (see[15]). Thus, in this case each K_{n+1} will have $\frac{(n-k+2)(n-k+1)}{2}$ bad edges. Thus, the total number of bad edges obtained from a $\delta^{(k)}$-coloring of $P_m \circ K_n$ is $\frac{m(n-k+2)(n-k+1}{2}$. □

Theorem 4.9. *For $C_m \circ P_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-coloring is given by*

$$b_2(C_m \circ P_n) = \begin{cases} \frac{3m(n-1)}{4}, & \text{if } m \text{ is even and for any } n, \\ \frac{(3m-1)(n-1)+4}{4}, & \text{if } m \text{ is odd and for any } n. \end{cases}$$

Proof. The minimum colors required in coloring $C_m \circ P_n$ is 3 and so the only value that k can take is 2. The following are the two cases discussed for $C_m \circ P_n$ when $k = 2$ for different parities of m when n is either even or odd.

Case 1: Let m be even. Now, every even cycle can be properly colored with two colors with $\frac{m}{2}$ possibility for each color, leading to no bad edge in the even cycle C_m. Now, the P_ns, corresponding to $\frac{m}{2}$ vertices of C_m that are assigned the color c_1, can be alternatively assigned the color c_2 and c_1 respectively, leading to a total of $\frac{m}{2} \lfloor \frac{n}{2} \rfloor$ bad edges between them (Note that, if the P_ns are alternatively colored with the colors c_1 and c_2 respectively, there will be $\lceil \frac{n}{2} \rceil$ vertices that receive the color c_1, leading to $\lceil \frac{n}{2} \rceil$ bad edges between the C_m and P_n). Now, the remaining P_ns, adjacent to the vertices of C_m which are assigned the color c_2, should be exclusively colored with the color c_1 to maintain the definition of $\delta^{(k)}$-coloring. This will lead to $\frac{m}{2}(n-1)$ bad edges. Thus, the minimum total number of bad edges resulting from $\delta^{(k)}$-coloring of $C_m \circ P_n$ when m is even and for any n is $\frac{m}{2} \lfloor \frac{n}{2} \rfloor + \frac{m}{2}(n-1) = \frac{3m(n-1)}{4}$.

Case 2: Consider m to be odd. We know that, $\delta^{(k)}$-coloring of an odd cycle with $k = 2$ available colors will lead to 1 bad edge in the cycle, with $\lceil \frac{m}{2} \rceil$ vertices receiving the color c_1 and $\lfloor \frac{m}{2} \rfloor$ vertices the color c_2. Now, as explained in the above case, the P_ns that are adjacent to the vertices that are assigned the color c_1 will lead to a total of $\lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor$ bad edges and the P_ns adjacent to the vertices that are colored with c_2 will lead to $\lfloor \frac{m}{2} \rfloor(n-1)$ bad edges. Thus, the minimum total number of bad edges obtained from $\delta^{(k)}$-coloring of $C_m \circ P_n$ when m is odd and for any n is $1 + \lceil \frac{m}{2} \rceil \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor(n-1) = \frac{(3m-1)(n-1)+4}{4}$. □

Theorem 4.10. *For $C_m \circ K_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-coloring where $1 \leq k \leq \chi(C_m \circ K_n) - 1$ is given by*

$$b_k(C_m \circ K_n) = \begin{cases} \frac{m(n-k+1)(n-k+2)}{2}, & \textit{if } m \textit{ is even and for all } n, \\ \frac{m(n-k+1)(n-k+2)+2}{2}, & \textit{if } m \textit{ is odd and for all } n. \end{cases}$$

Proof. The chromatic number of $C_m \circ K_m$, for any parity of m and n, is $n+1$. In this case, the k will be $2 \leq k \leq n$. For the different parities of m, there are two cases that are addressed separately as follows.

Case 1: Let m be even. Now, the minimum number of colors required to color an even cycle is 2. Hence, for any value of k, C_m will lead to no bad edges. Now, as explained in Theorem 4.9, every K_n adjacent to the $\frac{m}{2}$ vertices receiving the color c_1, can be colored with k colors leading to a total of $\frac{m}{2}\frac{(n-k+1)(n-k)}{2}$ bad edges in the K_ns. Also, there are $\frac{m}{2}(n-k+1)$ bad edges between these K_n and C_m. Now, the remaining K_n, adjacent to the $\frac{m}{2}$ vertices that are colored with a color other than c_1, say c_2, cannot be assigned with the color c_2 to meet the definition of $\delta^{(k)}$-coloring. Thus, these K_ns are colored with only $k-1$ colors, leading to a total of $\frac{m}{2}\frac{(n-k+2)(n-k+1)}{2}$ bad edges. Hence, the total number of bad edges resulting from $\delta^{(k)}$-coloring of $C_m \circ K_n$ when m is even and for any n is $\frac{m}{2}\frac{(n-k+1)(n-k)}{2} + \frac{m}{2}(n-k+1) + \frac{m}{2}\frac{(n-k+2)(n-k+1)}{2} = \frac{m(n-k+1)(n-k+2)}{2}$.

Case 2: Let m be odd. The minimum colors required to color an odd cycle is 3. Now, the values of k is $2 \leq k \leq n$. When $k \geq 3$, the C_m will lead to no bad edges. However, when $k = 2$, there will be an edge in C_m which is bad. A common $\delta^{(k)}$-coloring for both the cases is discussed as follows. Color the C_m with only 2 colors say c_1 and c_2. This will lead to 1 bad edge in C_m (see [15]). Now, the remaining K_ns, as explained in Theorem 4.9 and Case 1 of the current theorem, will lead to $\lceil \frac{m}{2} \rceil \frac{(n-k+1)(n-k)}{2}$ and $\lfloor \frac{m}{2} \rfloor \frac{(n-k+2)(n-k-1)}{2}$ bad edges in K_ns that are adjacent to the vertices that are colored with the colors c_1 and c_2 respectively in C_m. Now, between the vertices of C_ms and K_n's that receive the color c_1, there are $\lceil \frac{m}{2} \rceil(n-k+1)$ bad edges. Thus, the total number of bad edges resulting from $\delta^{(k)}$-coloring of $C_m \circ K_n$ when m is odd and for any n is $1 + \lceil \frac{m}{2} \rceil \frac{(n-k+1)(n-k)}{2} + \lfloor \frac{m}{2} \rfloor \frac{(n-k+2)(n-k-1)}{2} + \lceil \frac{m}{2} \rceil(n-k+1) = \frac{m(n-k+1)(n-k+2)+2}{2}$. Now, when $k \geq 3$, the odd cycle can be properly colored with $k = 3$ colors leading to no bad edge in the C_m. However, there will be a total of $\lceil \frac{m}{2} \rceil$ of vertices that receive the color other than the color c_1 and $\lfloor \frac{m}{2} \rfloor$ vertices that receive the color c_1. Now, this will lead to $\lceil \frac{m}{2} \rceil \frac{(n-k+2)(n-k+1)}{2}$ and $\lfloor \frac{m}{2} \rfloor \frac{(n-k+1)(n-k)}{2}$ bad edges between C_m and K_n. Now, when both the colorings are compared, the number of bad edges leading from either of the above mentioned $\delta^{(k)}$-coloring is the same. □

Theorem 4.11. *For $K_m \circ P_n$, the minimum number of bad edges obtained from the $\delta^{(k)}$-coloring for any m and n is given by*

$$b_k(K_m \circ P_n) = \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 3, \\ \frac{(m-1)(m-2)}{2} + (n-1) + (m-1)\lfloor \frac{n}{2} \rfloor, & \text{if } k = 2. \end{cases}$$

Proof. The chromatic number of $K_m \circ P_n$ is m. Thus, the available colors in this case is $2 \leq k \leq m-1$. There are two cases for two different values of k which are as explained below.

Case 1: Consider the case where $k \geq 3$. Now, it is known that, the minimum number of bad edges resulting from $\delta^{(k)}$-coloring of K_n is $\frac{(n-k+1)(n-k)}{2}$ (see [15]). Since the graph $K_m \circ P_n$ has a complete graph K_m as its induced subgraph, the minimum number of bad edges that $K_m \circ P_n$ will have is at least that of the $b_k(K_m)$. Now, we prove that, in this case, it is exactly $b_k(K_m)$. The minimum number of bad edges obtained from the $\delta^{(k)}$-coloring of K_m is $\frac{(m-k+1)(m-k)}{2}$. Now, since $k \geq 3$, the path graph P_n can be properly colored with any two colors other than the color assigned to its corresponding vertex of K_m. Thus, the minimum number of bad edges obtained from the $\delta^{(k)}$-coloring of $K_m \circ P_n$ is $\frac{(m-k+1)(m-k)}{2}$. A $\delta^{(k)}$-coloring of that explains the same is discussed as follows. Let $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ be the vertices of K_m and P_n respectively. Color the vertices $v_1, v_2, \ldots, v_k$ of the K_m with the colors $c_1, c_2, \ldots, c_k$. This is a proper coloring with k different colors. Now, the remaining vertices are assigned the color c_1 to maintain the requirements of $\delta^{(k)}$-coloring. Now, each of the P_n adjacent to each of its corresponding vertices of K_m are assigned the color other than the corresponding vertex. The P_n, adjacent to its corresponding vertex v_1 which is assigned the color c_1, can be properly colored with the two colors say c_2 and c_3. Similarly, the P_n, adjacent to the vertex v_2 that is assigned the color c_2, can be properly colored with two colors say c_1 and c_3. Thus, every mP_n's can be properly colored like wise. Hence, the minimum number of bad edges obtained from $\delta^{(k)}$-coloring of $K_m \circ P_n$ is $\frac{(m-k+1)(m-k)}{2}$ $\forall$ m and n.

Case 2: Let $k = 2$. Now, coloring K_m with $k = 2$ colors will lead to $\frac{(m-k+1)(m-k)}{2} = \frac{(m-1)(m-2)}{2}$ bad edges. This is because, only one vertex say the vertex v_1 can be assigned the color c_2 and all the remaining vertices must be assigned with color c_1, to maintain the conditions of $\delta^{(k)}$-coloring. Now, the P_n which is adjacent to the vertex v_1 should be colored with the color c_1, to meet the requirements of $\delta^{(k)}$-coloring. This will lead to $n-1$ bad edges in that particular P_n. The remaining $m-1$ P_n's, adjacent to the its corresponding vertices of K_m assigned the color c_1, can be alternatively colored with the colors c_2 and c_1 respectively (and not c_1 and c_2 respectively, as it will maximise the use of the color c_1 and maximise the number of bad edges between them). Thus, this will lead to $(m-1)\lfloor\frac{n}{2}\rfloor$ bad edges between them. Thus, there are a total of $\frac{(m-1)(m-2)}{2} + n - 1 + (m-1)\lfloor\frac{n}{2}\rfloor$ bad edges obtained from the $\delta^{(k)}$-coloring of $K_m \circ P_n$, when $k = 2$ and $\forall$ m and n. □

Theorem 4.12. *For $K_m \circ C_n$ for any m and n is even, the minimum number of bad edges obtained from the $\delta^{(k)}$-coloring is given by*

$$b_k(K_m \circ C_n) = \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 3, \\ \frac{m(m+n-3)+n+2}{2}, & \text{if } k = 2. \end{cases}$$

Proof. There are two different cases for a $\delta^{(k)}$-coloring of $K_m \circ C_n$ for different values of k and when n is even. Since $\chi(K_m \circ P_n) = m$, the values of k will lie between 1 and m. Considering all the above mentioned facts, both the cases are separately addressed as follows.

Case 1: Let $k \geq 3$. The proof explained in Case 1 of Theorem 4.11 applies to this case as well since both paths and even cycles are bipartite and can be properly colored with two colors by maintaining the constraints of $\delta^{(k)}$-coloring when $k \geq 3$. Thus, in this case the minimum number of bad edges resulting from $\delta^{(k)}$-coloring of $K_m \circ P_n$ is $\frac{(m-k+1)(m-k)}{2}$.

Case 2: Let $k = 2$. The proof for this case is similar to that of Case 2 of Theorem 4.11. The K_m will lead to $\frac{(m-2)(m-1)}{2}$ bad edges. Now, the only difference is that, the C_n that is adjacent to the vertex (only vertex) that is assigned the color c_2 is given the color c_1, leading to n bad edges in the cycle. All the remaining $m-1$ C_n's are assigned the color c_1 and c_2 alternatively leading to $(m-1)\frac{n}{2}$ bad edges between K_m and $(m-1)$ C_n's. Thus, the minimum total number of bad edges resulting from $\delta^{(k)}$-coloring of $K_m \circ C_n$ when n is even and $k = 2$ is $\frac{(m-2)(m-1)}{2} + (m-1)\frac{n}{2} + n = \frac{m(m+n-3)+n+2}{2}$. □

Theorem 4.13. *For $K_m \circ C_n$ for any m and n is odd, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is given by*

$$b_k(K_m \circ C_n) = \begin{cases} \frac{(m-k+1)(m-k)}{2}, & \text{if } k \geq 4, \\ \frac{(m-2)(m-1)+4}{2}, & \text{if } k = 3, \\ \frac{m(m+n)+n-1}{2}, & \text{if } k = 2. \end{cases}$$

Proof. The chromatic number of $K_m \circ C_n$ when n is odd is m and hence we discuss a $\delta^{(k)}$-coloring of the same for the different values of k where $2 \leq k \leq m-1$. There are three different cases for the same that are addressed separately as follows.

Case 1: Let $k \geq 4$. The minimum number of bad edges obtained from $\delta^{(k)}$-coloring of $K_m \circ C_n$ when $k \geq 4$ is $\frac{(m-k+1)(m-k)}{2}$. The proof for this case is same as that of the proof explained in Case 1 of Theorems 4.11 and 4.12.

Case 2: Let $k = 3$. The K_m when colored with $k = 3$ colors will lead to $\frac{(m-k+1)(m-k)}{2} = \frac{(m-2)(m-3)}{2}$ bad edges (see [15]). Now, there are only two vertices in K_m say v_1 and v_2 that can be colored with the colors c_2 and c_3. Rest of all the vertices have to be colored with the color c_1, to meet the requirements of a $\delta^{(k)}$-coloring. Now, since C_n is an odd cycle, it will require at least 3 colors to color it properly. Although, the number of available colors is 3, since these colors are used in the coloring of K_m, each cycle will lead a minimum of bad edges in the cycle or between the K_m and its corresponding C_n. Here, the vertex v_1 of K_m is assigned the color c_2 and hence the cycle is colored with two colors c_1 and c_3 leading to no bad edge between them. However, there will be a bad edge in C_n when colored with two colors (see [15]). Similarly, in the case of the vertex v_2 that is assigned the color c_3, its corresponding C_n will lead to 1 bad edge when colored with the colors c_1 and c_2. The remaining $(m-2)$ C_n's will lead to one bad edge between the vertices of K_m and its corresponding C_n. Thus, the total number of bad edges obtained from $\delta^{(k)}$-coloring of $K_m \circ C_n$ when n is odd is $\frac{(m-2)(m-3)}{2} + 2 + (m-2) = \frac{(m-2)(m-1)+4}{2}$. *Case 3*: Let $k = 2$. As explained in Case 2 of Theorem 4.12, only one vertex say v_1 of the K_m is given the color c_2, rest all are colored with the color c_1. This will lead to $\frac{(m-1)(m-2)}{2}$ bad edges. Now, C_n corresponding to the vertex v_1 is solely colored with c_1, to meets the requirements of $\delta^{(k)}$-coloring, and this leads in n bad edges in this particular cycle. The remaining C_ns are colored with two colors c_1 and c_2, leading to 1 bad edge in each of the $m-1$ C_ns and $(m-1)\lceil \frac{n}{2} \rceil$ bad edges between the K_m and C_n. Thus, the total number of bad edges resulting from $\delta^{(k)}$-coloring of $K_m \circ C_n$ when n is odd and $k = 2$ is $\frac{(m-1)(m-2)}{2} + (m-1) + (m-1)\lceil \frac{n}{2} \rceil + n = \frac{m(m+n)+n-1}{2}$. □

Theorem 4.14. *For $K_m \circ K_n$ for any m and n, the minimum number of bad edges obtained from a $\delta^{(k)}$-coloring is given by*

$$b_k(K_m \circ K_n) = \begin{cases} (m-k+1)(\frac{m-k}{2} + n - k + 1(\frac{n-k+2}{2}(\frac{m}{m-k+1}))), & \text{if } k \geq 3, \\ \frac{(n(n-3)+m)(m-1)+2n}{2}, & \text{if } k = 2. \end{cases}$$

Proof. We know that, the minimum number of bad edges in K_m resulting from $\delta^{(k)}$-coloring when the available colors are k, is $\frac{(m-k+1)(m-k)}{2}$. Each of the K_n's corresponding to the each of the vertex assigned the color c_1 in K_m will lead to $\frac{(n-k+1)(n-k)}{2}$ bad edges and between them there will be $(m-k+1)(n-k+1)$ bad edges (for a detailed explanation on the coloring pattern of $\delta^{(k)}$-coloring of complete graphs see [15, 4]). Now, the K_n's corresponding to the vertices that receive the color other than c_1 in K_m, i.e. the $k-1$ vertices, can be colored with $k-1$ colors only (the color assigned to its corresponding vertex in K_m, cannot be used in coloring its corresponding K_n). Thus, this will lead to $(k-1)\frac{(n-k+1)(n-k)}{2}$ bad edges between them. Thus, the total number of bad edges resulting from $\delta^{(k)}$-coloring of $K_m \circ K_n$ when $k \geq 3$ is $\frac{(m-k+1)(m-k)}{2} + (m-k+1)\frac{(n-k+1)(n-k)}{2} + (m-k+1)(n-k+1) + (k-1)\frac{(n-k+1)(n-k)}{2} = (m-k+1)(\frac{m-k}{2} + n-k+1(\frac{n-k+2}{2}(\frac{m}{m-k+1})))$. *Case 2*: Let $k = 2$. Coloring K_m with $k = 2$ colors will lead to $\frac{(m-k+1)(m-k)}{2} = \frac{(m-1)(m-2)}{2}$ bad edges. Now, all the corresponding K_ns, other than the one which is adjacent to the vertex assigned the color c_2 of K_m, are colored with $k = 2$ colors, leading to $\frac{(m-k+1)(n-k+1)(n-k)}{2} = \frac{(m-1)(n-1)(n-2)}{2}$ bad edges in the K_m. Now, between the vertices of K_m that are assigned the color c_1 i.e. $m-k+1 = m-1$ vertices of K_m and $m-k+1 = m-1$ K_ns there are $(m-k+1)(n-k+1) = (m-1)(n-1)$ bad edges. Now, the K_n adjacent to the vertex colored with the color c_2 of K_m should be given only the color c_1 to maintain the requirements of $\delta^{(k)}$-coloring, leading to $\frac{n(n-1)}{2}$ bad edges. Thus, the total number of bad edges resulting from $\delta^{(k)}$-coloring of $K_m \circ K_n$ when $k = 2$ is $\frac{(m-1)(m-2)}{2} + \frac{(m-1)(n-1)(n-2)}{2} + (m-1)(n-1) + \frac{n(n-1)}{2} = \frac{(n(n-3)+m)(m-1)+2n}{2}$. □

5 Conclusion

This paper focuses on a $\delta^{(k)}$-coloring of certain graph products viz. direct product of two graphs and corona product of two graphs. The graph classes that are discussed here are path P_n, cycle C_n and complete graph K_n with their different combinations depending on the commutative property of the products discussed. a $\delta^{(k)}$-coloring of different products can also be investigated. We have only relaxed one color class to have adjacency between the elements in it. However, permitting few more color classes to be non independent set to minimise the bad edges resulting from it can be a ground for further research. A comparative study on the number of bad edges obtained when one color class and more than one color are relaxed can also be a study of great research.

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Merlin Thomas Ellumkalayil
Department of Mathematics, CHRIST (Deemed to be University),
Bangalore-560029, Karnataka, INDIA
ellumkalayil.thomas@res.christuniversity.in

Sudev Naduvath
Department of Mathematics, CHRIST (Deemed to be University),
Bangalore-560029, Karnataka, INDIA
sudev.nk@christuniversity.in